A NOTE ON INVARIANT FINITELY ADDITIVE MEASURES

S. G. DANI¹

ABSTRACT. We show that under certain general conditions any finitely additive measure which is defined for all subsets of a set X and is invariant under the action of a group G acting on X is concentrated on a G -invariant subset Y on which the G -action factors to that of an amenable group. The result is then applied to prove a conjecture of S. Wagon about finitely additive measures on spheres.

It is well known that if G is an amenable group acting on a set X then there exist plenty of G -invariant finitely additive probability measures on $(X, \mathfrak{P}(X))$ where $\mathfrak{P}(X)$ is the class of all subsets of X (cf. [3] for details). However, such measures may fail to exist when G is nonamenable. In [6] S. Wagon conjectured that if G is a group of isometries of S^n , the n -dimensional sphere, such that for any G -invariant subset Y , the group $\{g/Y \mid g \in G\}$ of restrictions of elements of g to Y is nonamenable, then there does not exist any G -invariant finitely additive probability measure on $(S^n, \mathfrak{P}(S^n))$.

In this note we establish the above-mentioned conjecture. Further, we formulate a condition on actions of (abstract) groups, involving the isotropy subgroups and fixed point sets, which implies similar assertions in a more general situation (cf. Theorem 1.1). The condition holds for actions of subgroups G of any compact Lie group $\mathfrak{G}$ acting on homogeneous spaces of $\mathfrak{G}$. It also holds for actions of subgroups G of algebraic $\mathbf{R}$ -groups $\mathfrak{G}$ acting on homogeneous spaces of $\mathfrak{G}$ by algebraic $\mathbf{R}$ -subgroups. Thus, in all these cases we are able to conclude that G -invariant finitely additive probability measures (defined for all subsets) are concentrated on invariant sets on which the action factors to that of an amenable quotient of G (cf. §2).

A particular consequence is that if G is a nonamenable subgroup of $\mathrm{GL}(n+1, \mathbf{R})$ acting irreducibly on $\mathbf{R}^{n+1}$, then for the natural G -actions on $\mathbf{R}^{n+1} - (0)$, S^n or $\mathbf{P}^n$, there exist no invariant finitely additive probability measures (cf. Corollaries 2.2 and 2.4). We recall that by a theorem of A. Tarski [5] this is equivalent to existence of paradoxical decompositions for the action (cf. [6 and 7] for motivation and some results in that direction).

In some of the G -actions discussed above, e.g. G a group of isometries of S^n or G a subgroup of a compact Lie group $\mathfrak{G}$ acting on a homogeneous space of $\mathfrak{G}$, there exists a natural countably additive G -invariant probability measure defined on the class of Borel subsets. We prove that the measure extends to a G -invariant finitely additive measure defined on all subsets if and only if G is amenable (cf. Theorem

Received by the editors January 10, 1984.

1980 *Mathematics Subject Classification*. Primary 28A70.

Key words and phrases. Invariant finitely additive measures, amenable groups.

¹Supported in part by NSF MCS-8108814 (A02).

1.4 for a general result). For the case of groups of isometries of S^n this was first proved by S. Wagon [6].

In [1] the author obtained results analogous to those discussed above for the case of subgroups of $GL(n, \mathbf{Z})$ acting on $\mathbf{T}^n$ as group automorphisms. The present method is analogous but much simpler. In the case of the action on $\mathbf{T}^n$ the condition of Theorem 1.1 is not satisfied and consequently the proofs depend on various specific features of $\mathbf{T}^n$.

1. Main results. Let G be a group acting on a set X . For $x \in X$ we denote by G_x the isotropy subgroup under the G -action; viz. $G_x = \{g \in G \mid gx = x\}$. For any subgroup H of G we denote by F_H the set of fixed points of H ; that is $F_H = \{x \in X \mid hx = x \text{ for all } h \in H\}$.

For any set E we denote by $\mathfrak{P}(E)$ the class of all subsets of E . A (possibly empty) subclass $\mathfrak{C}$ of $\mathfrak{P}(E)$ is said to be of *type* $\mathfrak{L}$ if the following holds: for any family $\{E_\alpha\}_{\alpha \in \Lambda}$, where Λ is an indexing set and $E_\alpha \in \mathfrak{C}$ for all $\alpha \in \Lambda$, there exist $k \geq 1$ and $\alpha_1, \alpha_2, \dots, \alpha_k \in \Lambda$ such that $\bigcap_{\alpha \in \Lambda} E_\alpha = \bigcap_{i=1}^k E_{\alpha_i}$.

THEOREM 1.1. *Let G be a group acting on a set X . Suppose $\{G_x \mid x \in X\}$ and $\{F_H \mid H \text{ a subgroup of } G\}$ are of type $\mathfrak{L}$. Let μ be a G -invariant finitely additive probability measure on $(X, \mathfrak{P}(X))$. Then there exists a normal subgroup Q of G such that G/Q is amenable and $\mu(F_Q) = 1$.*

We need the following lemma from [4] (cf. Proposition 3.5 of [4]).

LEMMA 1.2. *Let G be a group acting on a set X . Suppose that for all $x \in X$ the isotropy subgroup G_x is amenable. Suppose also that there exists a G -invariant finitely additive probability measure μ on $(X, \mathfrak{P}(X))$. Then G is amenable.*

We first prove the following.

PROPOSITION 1.3. *Let G be a group acting on a set X . Suppose $\{G_x \mid x \in X\}$ is of type $\mathfrak{L}$. Suppose also that there exists a G -invariant finitely additive probability measure μ such that for any nontrivial subgroup H of G , $\mu(F_H) < 1$. Then G is amenable.*

PROOF. If possible let G be nonamenable. Let $\mathcal{N}$ be the class of all nonamenable subgroups S of G such that either $S = G$ or $S = \bigcap_{i=1}^k G_{x_i}$, where $k \geq 1$ and $x_1, \dots, x_k \in X$. Let $\mathcal{M}$ be the subset of $\mathcal{N}$ consisting of minimal elements of $\mathcal{N}$; that is $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $N \subset M$ implies $N = M$. Since the class of subgroups $\{G_x \mid x \in X\}$ is of type $\mathfrak{L}$, it follows that $\mathcal{M}$ is nonempty and that in fact any $N \in \mathcal{N}$ contains a minimal element (otherwise there would exist a sequence $x_1, \dots, x_k, \dots$, in X such that $\bigcap_{i=1}^n G_{x_i}$ is a strictly decreasing sequence).

Let $M \in \mathcal{M}$. Being nonamenable, in particular M is nontrivial. Hence, by hypothesis, $\mu(F_M) < 1$ or, equivalently, $\mu(X - F_M) > 0$. In particular, $X - F_M$ is nonempty. Consider the M -action on $X - F_M$. The isotropy subgroup of any $x \in X - F_M$ under the M -action is $M \cap G_x$ and it is a proper subgroup of M . We note that $M \cap G_x$ is amenable for all $x \in X$; if not, $M \cap G_x \in \mathcal{N}$, which would mean M is not minimal in $\mathcal{N}$. On the other hand, on $X - F_M$ we have an M -invariant finitely additive probability measure ν defined by $\nu(E) = \mu(E)/\mu(X - F_M)$ for all $E \subset X - F_M$. By Lemma 1.2 these observations imply that M is amenable—a contradiction. Hence G must be amenable.

PROOF OF THE THEOREM. Consider the class of subgroups $\mathfrak{C} = \{H \mid \mu(F_H) = 1\}$. Let $F = \bigcap \{F_H \mid H \in \mathfrak{C}\}$. Since $\{F_H \mid H \subset G\}$ is of type $\mathcal{L}$ there exist $k \geq 1$ subgroups $H_1, \dots, H_k \in \mathfrak{C}$ such that $F = \bigcap_{i=1}^k F_{H_i}$. Then $\mu(F) = 1 - \mu(X - F) \geq 1 - \sum \mu(X - F_{H_i}) = 1$. Let Q be the subgroup generated by $\bigcup \{H \mid H \in \mathfrak{C}\}$. Since μ is G -invariant, $\mathfrak{C}$ is invariant under conjugation and consequently Q is a normal subgroup of G . Further, clearly $F_Q = F$ so that $\mu(F_Q) = 1$ and hence Q is the unique maximal element of $\mathfrak{C}$. To complete the proof, we need only show that G/Q is amenable. F_Q is a G -invariant set and the action of G on F_Q factors to an action of G/Q . The restriction of μ to F_Q is a G/Q -invariant finitely additive probability measure on F_Q . It is evident that the G/Q -action on F_Q satisfies the conditions of Proposition 1.3 (G/Q and F_Q in the place of G and X respectively). While the isotropy subgroups form a class of type $\mathcal{L}$ because of the corresponding property for the G -action, the set of fixed points of any nontrivial subgroup has μ -measure < 1 because of our choice of Q as the unique maximal element of $\mathfrak{C}$ as defined above. Hence, by Proposition 1.3, G/Q is amenable, which proves the theorem.

Let $(X, \mathfrak{M}, m)$ be a measure space; that is, $\mathfrak{M}$ is a σ -algebra of subsets of X , and m is a (countably additive) measure defined on $\mathfrak{M}$. Let G be a group acting on X preserving $\mathfrak{M}$ and m ; that is, for all $g \in G$ and $E \in \mathfrak{M}$, $gE \in \mathfrak{M}$ and $m(gE) = m(E)$. The G -action is said to be *weakly measurable* if for any subgroup H of G , $F_H \in \mathfrak{M}$. A weakly measurable action is said to be *essentially factorable* through a normal subgroup H of G if $m(X - F_H) = 0$. A finitely additive measure μ on $(X, \mathfrak{P}(X))$ is said to extend the measure m on $(X, \mathfrak{M})$ if $\mu(E) = m(E)$ for all $E \in \mathfrak{M}$.

THEOREM 1.4. *Let G be a group acting on a measure space $(X, \mathfrak{M}, m)$, preserving $\mathfrak{M}$ and m . Suppose $m(X) = 1$ and the G -action is weakly measurable. Suppose also that $\{G_x \mid x \in X\}$ and $\{F_H \mid H \text{ a subgroup of } G\}$ are of type $\mathcal{L}$. Then there exists a G -invariant finitely additive measure μ on $(X, \mathfrak{P}(X))$ extending the measure m on $(X, \mathfrak{M})$ if and only if the G -action is essentially factorable through a normal subgroup Q such that G/Q is amenable.*

PROOF. Suppose there exists a G -invariant finitely additive measure μ on $(X, \mathfrak{P}(X))$ extending m . By Theorem 1.1 there exists a normal subgroup Q of G such that G/Q is amenable and $\mu(F_Q) = 1$. Since $F_Q \in \mathfrak{M}$ and μ extends m , we get $m(F_Q) = 1$, that is, the action is essentially factorable through Q .

Conversely, suppose the action is essentially factorable through a normal subgroup Q such that G/Q is amenable. By a well-known result (cf. [3, Theorem 5.1]) the measure m_Q on $(F_Q, \mathfrak{M}_Q)$, where $\mathfrak{M}_Q = \{E \cap F_Q \mid E \in \mathfrak{M}\}$, defined by $m_Q(E) = m(E)$ for all $E \in \mathfrak{M}_Q$ extends to a G/Q -invariant (under the factor action) finitely additive measure μ_Q on $(F_Q, \mathfrak{P}(F_Q))$. Put $\mu(E) = \mu_Q(E \cap F_Q)$, for all $E \in \mathfrak{P}(X)$. Then μ is a G -invariant finitely additive measure on $(X, \mathfrak{P}(X))$ extending the measure m on $(X, \mathfrak{M})$.

2. Examples. We now apply Theorems 1.1 and 1.4 to various situations.

(i) Let $X = S^n$, the n -dimensional sphere (of unit vectors in $\mathbf{R}^{n+1}$, with respect to the usual norm) and let G be a subgroup of $O(n+1)$, the orthogonal group, acting as isometries of S^n . For each $x \in S^n$, $G_x = G \cap O(n+1)_x$. Since evidently $\{O(n+1)_x \mid x \in S^n\}$ is evidently of type $\mathcal{L}$, so is $\{G_x \mid x \in S^n\}$. On the other hand, for each subgroup $H \subset G \subset O(n+1)$, F_H is a compact differentiable submanifold

with finitely many connected components. It follows therefore that $\{F_H \mid H \text{ a subgroup of } G\}$ is of type $\mathfrak{L}$. Hence we get the following.

COROLLARY 2.1. (a) *Let μ be a G -invariant finitely additive probability (or finite) measure on $(S^n, \mathfrak{P}(S^n))$. Then there exists a normal subgroup Q of G such that G/Q is amenable and $\mu(S^n - Y) = 0$, where $Y = F_Q = \{x \in S^n \mid qx = x \text{ for all } q \in Q\}$. Y is a G -invariant set and $\{g/Y \mid g \in G\}$, the group of restrictions of elements of G to Y , is an amenable group.*

(b) *There exists a G -invariant finitely additive measure μ on $(S^n, \mathfrak{P}(S^n))$ extending the standard ($O(n+1)$ -invariant) measure m on $(S^n, \mathfrak{B})$, where $\mathfrak{B}$ is the σ -algebra of Borel sets, if and only if G is amenable.*

PROOF. (a) Let Q be as given by Theorem 1.1. Then since Q is normal, $Y = F_Q$ is G -invariant. Since Q fixes each $y \in Y$ the group $\{g/Y \mid g \in G\}$ is a quotient of G/Q and therefore amenable (cf. [2]).

(b) If such an extension exists, then by Theorem 1.4 there exists a normal subgroup Q such that G/Q is amenable and $m(S^n - F_Q) = 0$. The latter condition implies that F_Q is dense in S^n and, since it is closed, we have $F_Q = S^n$. But identity is the only isometry fixing all points. Hence Q is trivial and consequently G is amenable. Conversely, if G is amenable, then by a well-known result (cf. Theorem 5.1 of [3]) there exists a G -invariant finitely additive measure on $(S^n, \mathfrak{P}(S^n))$ extending m .

Part (b) was proved earlier by Wagon (cf. [6]). He also noted part (a) in the particular case $n = 2$ and conjectured its validity for all n (cf. [6]); the corollary establishes the conjecture. (We should, however, note the following: In the statement of the conjecture in [6, p. 81], it is not explicit whether the measure is meant to be finite. But in the particular case of S^2 which is settled in [6] and from which the author motivates the conjecture, the measure is implicitly assumed to be finite (cf. Theorems 6 and 7 and Proposition 1 in [6]). Whether the conjecture is true for σ -finite additive measures is not clear, even for S^2 .)

In [6] it is actually shown (cf. Theorem 7 of [6]) that for any nonamenable group G of isometries of S^2 there does not exist any (finite) G -invariant finitely additive measure on $(S^2, \mathfrak{P}(S^2))$. We note that if G is a nonamenable subgroup of $O(3)$ then the natural action of G on $\mathbf{R}^3$ is irreducible (that is, there is no proper nontrivial invariant subspace). The above-mentioned assertion from [6] generalizes to the following.

COROLLARY 2.2. *Let G be a nonamenable subgroup of $O(n+1)$ acting irreducibly on $\mathbf{R}^{n+1}$. Then there does not exist any G -invariant finitely additive probability measure on $(S^n, \mathfrak{P}(S^n))$.*

PROOF. If possible let μ be such a finitely additive measure. By Corollary 2.1 there exists a normal subgroup Q of G such that G/Q is amenable and $\mu(F_Q) = 1$. Evidently, $F_Q = S^n \cap W$, where $W = \{x \in \mathbf{R}^{n+1} \mid qx = x \text{ for all } q \in Q\}$. Since Q is normal W is a G -invariant subspace of $\mathbf{R}^{n+1}$. By irreducibility either $W = (0)$ or $\mathbf{R}^{n+1}$. The former is impossible since F_Q would then be empty, while actually $\mu(F_Q) > 0$. On the other hand, if $W = \mathbf{R}^{n+1}$ then $F_Q = S^n$, which means Q must be the trivial subgroup; but in that case G must be amenable—a contradiction to the hypothesis. Hence there cannot exist such a finitely additive invariant measure.

(ii) The above examples generalize in a natural way to homogeneous spaces of compact Lie groups. Let $\mathfrak{G}$ be a compact Lie group and $\mathfrak{S}$ be a closed subgroup of $\mathfrak{G}$. Let $X = \mathfrak{G}/\mathfrak{S}$ and let G be a subgroup of $\mathfrak{G}$ acting on X on the left. Then arguments as above yield the following.

COROLLARY 2.3. (a) *Let μ be a G -invariant finitely additive probability measure on $(X, \mathfrak{P}(X))$. Then there exists a normal subgroup Q of G such that G/Q is amenable and $\mu(X - F_Q) = 0$.*

(b) *There exists a G -invariant finitely additive measure μ on $(X, \mathfrak{P}(X))$ extending the $\mathfrak{G}$ -invariant (countably additive) probability measure on $(X, \mathfrak{B})$, where $\mathfrak{B}$ is the σ -algebra of Borel sets, if and only if G is amenable.*

(iii) Similar arguments may be made for homogeneous spaces of algebraic groups: Let $\mathfrak{G}$ be an algebraic $\mathbf{R}$ -group (the group of $\mathbf{R}$ -elements of an algebraic group defined over $\mathbf{R}$), e.g. $\mathrm{GL}(n, \mathbf{R})$. Let $\mathfrak{S}$ be an algebraic $\mathbf{R}$ -subgroup of $\mathfrak{G}$. Let $X = \mathfrak{G}/\mathfrak{S}$ and let G be a subgroup of $\mathfrak{G}$ acting on X on the left. Then for each $x \in X$, $G_x = G \cap \mathfrak{G}_x$. $\{\mathfrak{G}_x \mid x \in X\}$ is a family of algebraic $\mathbf{R}$ -subgroups and therefore it is of type $\mathfrak{L}$. Hence so is $\{G_x \mid x \in X\}$. On the other hand, for any subgroup H , F_H is the set of $\mathbf{R}$ -elements of an algebraic variety. Therefore $\{F_H \mid H \text{ a subgroup of } G\}$ is of type $\mathfrak{L}$. It follows that Corollary 2.2(a) holds verbatim for G and X as above. In general, $(X, \mathfrak{B})$, where $\mathfrak{B}$ is the Borel σ -algebra, may not carry any finite G -invariant (countably additive) measure. If it does then Corollary 2.2(b) also holds verbatim for those G and X . We note that since compact Lie groups are algebraic $\mathbf{R}$ -groups these results generalise Corollary 2.3. Similarly the following generalises Corollary 2.1.

COROLLARY 2.4. *Let G be a subgroup of $\mathrm{GL}(n, \mathbf{R})$. Let X be either $\mathbf{R}^n - (0)$ or S^{n-1} or $\mathbf{P}^{n-1}$ ($(n-1)$ -dimensional projective space) and consider the natural G -action on X . Let μ be a G -invariant finitely additive probability measure on $(X, \mathfrak{P}(X))$. Then there exists a normal subgroup Q of G such that G/Q is amenable and $\mu(F_Q) = 1$. In particular, if G is nonamenable, and the action of G on $\mathbf{R}^n$ is irreducible (e.g. if $G = \mathrm{SL}(n, \mathbf{Z})$) then there does not exist any G -invariant finitely additive probability (or finite) measure on $(X, \mathfrak{P}(X))$.*

In the case of $X = \mathbf{R}^n - (0)$ or $\mathbf{P}^{n-1}$ the isotropy subgroups for the (transitive) $\mathrm{GL}(n, \mathbf{R})$ -action are algebraic $\mathbf{R}$ -subgroups and the result follows from the above remarks. For S^{n-1} it may be deduced either from the result for $\mathbf{P}^{n-1}$ or directly from Theorem 1.1. If the action of G on $\mathbf{R}^n$ is irreducible, then as in the proof of Corollary 2.2 we see that for any normal subgroup H of G , F_H must be either empty or X (for any X as above). Hence for such a G there cannot exist any invariant finitely additive measure on $(X, \mathfrak{P}(X))$ unless it is amenable.

In [1] we proved the analogous assertion for the actions of subgroups of $\mathrm{GL}(n, \mathbf{Z})$ acting on $\mathbf{T}^n$, the n -dimensional torus, as the group of automorphisms. In that case, however, Theorems 1.1 and 1.4 do not apply since G_x for $x \in \mathbf{T}^n$ cannot in general be expressed as $G \cap \mathfrak{G}_x$ for some algebraic group $\mathfrak{G}$; in fact $\{G_x \mid x \in \mathbf{T}^n\}$ is not of type $\mathfrak{L}$. The proof there uses, together with the ideas underlying the proof of Theorem 1.1, certain specific features of the torus.

REFERENCES

1. S. G. Dani, *On invariant finitely additive measures for automorphism groups of tori*, Trans. Amer. Math. Soc. (to appear).
2. F. P. Greenleaf, *Invariant means on topological groups*, Van Nostrand Reinhold, New York, 1969.
3. J. Mycielski, *Finitely additive invariant measures*. I, Colloq. Math. **42** (1979), 309–318.
4. J. Rosenblatt, *Uniqueness of invariant means for measure-preserving transformations*, Trans. Amer. Math. Soc. **265** (1981), 623–636.
5. A. Tarski, *Algebraische Fassung des Massproblems*, Fund. Math. **31** (1938), 47–66.
6. S. Wagon, *Invariance properties of finitely additive measures on $\mathbf{R}^n$* , Illinois J. Math. **25** (1981), 74–86.
7. ———, *The use of shears to construct paradoxes in $\mathbf{R}^2$* , Proc. Amer. Math. Soc. **85** (1982), 353–359.

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

Current address: School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India