ON SPHERICAL SUBMANIFOLDS WITH NULLITY

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ABSTRACT. We derive a new condition for spherical submanifolds with positive relative nullity. As an application we obtain that the Ricci curvature is bounded from above by the curvature of the ambient space.

In this note we improve earlier results of O'Neill-Stiel [O'N-S], Ferus [F], Abe [A], and Rodriguez [R] on complete submanifolds with positive relative nullity in euclidean spheres; cf. [R] for the basic facts and references.

THEOREM 1. *Let $f: M^n \rightarrow S_c^{n+p}$ be an isometric immersion of a complete manifold in a sphere of curvature c with relative nullity $\nu > 0$ everywhere. Suppose there exists a point x in the open set where ν is minimal so that the (normalized) Ricci curvature satisfies*

$$(1) \quad \text{Ric}(x) \geq c.$$

Then f is totally geodesic.

In [A] and [R] the same conclusion was obtained under the hypothesis that (1) holds everywhere. As an immediate consequence of our theorem, we have that if $\nu > 0$ is constant and f is not totally geodesic, then $\text{Ric} < c$ everywhere. Examples for this situation are the minimal (homogeneous) isoparametric hypersurfaces with three distinct principal curvatures (cf. [M]). The completeness assumption above can be weakened (cf. the Remark later).

Theorem 1 will be a consequence of the following much stronger result.

THEOREM 2. *Let $f: M^n \rightarrow S_c^{n+p}$ be an isometric immersion, M^n complete, $\nu > 0$ everywhere. Then the numbers of positive and negative principal curvatures in each normal direction are equal at any point where $\nu = \nu_0$ is minimal.*

We need the following

LEMMA. *Let Q^{n+p} be a space of constant curvature and $f: M^n \rightarrow Q^{n+p}$ an isometric immersion with constant relative nullity. Then, for any parallel normal field ξ of M along a leaf L of the nullity foliation, the rank of the second fundamental form A_ξ is constant along L .*

PROOF. Fix a leaf L . Locally in M , extend ξ , and let V, W be vector fields so that V is tangent and W is normal to the leaves, and W is parallel along L . By the Codazzi equations,

$$\nabla_V A_\xi W = \nabla_W A_\xi V - A_{\nabla_W^\perp \xi} V + A_{\nabla_V^\perp \xi} W + A_\xi \nabla_V W - A_\xi \nabla_W V,$$

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where $\nabla, \nabla^\perp$ are the tangent and normal connections of M . Along L the first four terms vanish. As in [R] we consider the tensor B on the normal bundle of L in M , where BW is the normal component of $-\nabla_W V$. Thus, $\nabla_V A_\xi = A_\xi B$, or

$$(2) \quad \nabla_V A_\xi = B^t A_\xi.$$

Here we view A_ξ as being restricted to the normal bundle of L . By (2) A_ξ satisfies a linear differential equation along any integral curve of V in L , which implies that the kernel of A_ξ is parallel and, therefore, the rank of A_ξ is constant.

PROOF OF THEOREM 2. Consider the maximal set U in M where $\nu = \nu_0$. It is well known that (since M is complete) U is foliated by complete totally geodesic leaves which f maps onto great spheres. Thus the antipodal map $I = -\text{id}$ of S^{n+p} induces an involution I on U . Fix $x \in U$, a normal vector ξ of M at x , and a leaf L through x in U . We can view ξ as a parallel field along L . Note that any parallel normal field of M in S^{n+p} is constant along each leaf in euclidean space $\mathbf{R}^{n+p+1}$. Now we have for the second fundamental form, $I_* A_\xi = A_{I_* \xi} I_*$, so

$$(3) \quad A_{\xi_x} = A_{-\xi_{-x}} = -A_{\xi_{-x}}.$$

Here we identify the tangent spaces of M at x and $-x$ by parallel transport in euclidean space. By the Lemma A_ξ has constant rank on the sphere L , so the number of positive eigenvalues of A_{ξ_x} is equal to the number of positive eigenvalues of $A_{\xi_{-x}}$. Now our claim follows from (3).

REMARK. It is clear from the proof that the completeness of M can be replaced by the weaker assumption that a leaf L contains at least one pair of antipodal points. Then the conclusion of the theorem holds for all $x \in L$.

PROOF OF THEOREM 1. It is a simple algebraic fact that $\text{Ric}(x) \geq c$ for $x \in U$ implies A_ξ is positive definite on the normal space of the leaf L_x in M for some $\xi \neq 0$ (cf. [R, p. 183]). This contradicts Theorem 2, unless $\nu_0 = n$, i.e. f is totally geodesic.

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