THE TWO MULTIPLICATIONS ON BU

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ABSTRACT. We prove that BU_p has only two standard H -structures.

The two most familiar H -multiplications on BU arise from the direct sum of complex vector bundles and from the tensor product of virtual line bundles; BU with these multiplications will be denoted by $BU^\oplus$ and $BU^\otimes$ respectively [2]. Both multiplications are standard, define a standard H -space (X, e, μ) to be an H -space whose rational Pontrjagin ring $H_*(X, Q)$ induced by μ is both associative and graded commutative. We consider H -spaces localized at a fixed prime p .

THEOREM 1.1. *Let m be a standard H -multiplication on BU . Then BU_p^m is H -equivalent to $BU_p^\oplus$ or $BU_p^\otimes$.*

We recall that two H -spaces (X, e, μ) and (Y, f, ν) are H -equivalent if there exists a homotopy equivalence $k: X \rightarrow Y$ such that $\nu(k \times k) \simeq k\mu$. The two classes of multiplications on BU_p in Theorem 1.1 can easily be distinguished for the Frobenius map

$$\xi: H_2(BU^m, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_{2p}(BU^m, \mathbb{Z}/p\mathbb{Z})$$

defined by $\xi(x) = x^p$ is nontrivial when $m = \oplus$ but is the zero homomorphism if $m = \otimes$. Theorem 1.1 completes results about standard multiplications on BU_p proved in [4]. The strongest result in this context was given in §4 of [4]. X is assumed to have the homotopy type of a connected CW-complex with finite skeleta.

THEOREM 1.2. *Let X be a standard H -space and p an odd prime. Assume that*

- (a) $H^*(X, \mathbb{Z}/p\mathbb{Z})$ is a polynomial algebra,
 - (b) $\dim QH^{2i}(X, \mathbb{Z}/p\mathbb{Z}) \leq 1$ for all i and the equality holds at least for $1 \leq i \leq p-1$,
 - (c) $\xi: H_2(X, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_{2p}(X, \mathbb{Z}/p\mathbb{Z})$ is nontrivial.
- Then $X_p \simeq BU_p^\oplus$ as an H -space.*

We can now add a companion result for $BU_p^\otimes$.

THEOREM 1.3. *Let X be a standard H -space and p an odd prime. Assume that*

- (a) $H^*(X, \mathbb{Z}/p\mathbb{Z})$ is a polynomial algebra,
- (b') $\dim QH^{2i}(X, \mathbb{Z}/p\mathbb{Z}) \leq 1$ for all i and the equality holds at least for $1 \leq i \leq p$,

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(b)'' $P^1: QH^{2p}(X, Z/pZ) \rightarrow QH^{4p-2}(X, Z/pZ)$ is an isomorphism,

(c) $\xi: H_2(X, Z/pZ) \rightarrow H_{2p}(X, Z/pZ)$ is trivial.

Then $X_p \simeq BU_p^{\oplus}$ as an H -space.

It was shown in Theorem 4.4 of [4] that the hypotheses of Theorem 1.3 imply that X_p and BU_p have the same homotopy type. So it remains to show that we can choose a homotopy equivalence which is an H -map. At odd primes Theorem 1.2 and Theorem 1.3 imply Theorem 1.1. When $p = 2$ and the Frobenius map is nontrivial the conclusion of Theorem 1.1 follows from the final few lines of [4]. Therefore in order to complete the proofs of both Theorem 1.1 and Theorem 1.3 it remains to show that if p is any prime $X_p \simeq BU_p$ and the Frobenius map

$$\xi: H_2(X, Z/pZ) \rightarrow H_{2p}(X, Z/pZ)$$

is trivial, then $X_p \simeq BU_p^{\oplus}$ as an H -space.

The results of this paper can be extended to the classifying spaces of other stable classical groups except that when $p = 2$ we must exclude BO and BSO as the central result used in the proofs is Theorem 3.1 of [4] and this cannot be used when homology 2-torsion is present. Therefore we state results for BSU and BSp only.

THEOREM 1.4. *Let m be a standard multiplication on BG , where $G = SU$ or Sp . Then $BG_p^m \simeq BG_p^{\oplus}$ as an H -space.*

The result corresponding to Theorems 1.2 and 1.3 for BSp is

THEOREM 1.5. *Let X be a standard H -space and p an odd prime. Assume that*

(a) $H^*(X, Z/pZ)$ is a polynomial algebra,

(b)' $\dim QH^{4i+2}(X, Z/pZ) = 0$ for all i ,

(b)'' $\dim QH^{4i}(X, Z/pZ) \leq 1$ for all i and the equality holds at least for $1 \leq i \leq (p-1)/2$.

Then $X_p \simeq BSp_p^{\oplus}$ as an H -space.

A similar theorem can be formulated for BSU .

Again most of the proofs of Theorems 1.4 and 1.5 are in [4]. If $G = SU$ and p is odd, Theorem 1.4 follows from Theorem 4.5 of [4] and if $p = 2$ the result follows as in the last part of the proof for odd p . Theorem 1.5 follows from Theorem 4.1 of [4]. Theorem 1.4 for $G = Sp$ and p odd follows from Theorem 1.5. Therefore to complete the proof of Theorem 1.4 we must show that if $X_2 \simeq BSp_2$, then $X_2 \simeq BSp_2^{\oplus}$ as an H -space.

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2. The proofs of Theorems 1.1 and 1.3. We assume that $X \simeq BU$ at the prime p and that the Frobenius map

$$\xi: H_2(X, Z/pZ) \rightarrow H_{2p}(X, Z/pZ)$$

is trivial. We must define an H -map $k: X_p \rightarrow BU_p^{\oplus}$ which is a homotopy equivalence. The key to the proof is Proposition 2.1 which is established using the main

theorem of [5]. We recall that $\Gamma(x, r) \subset Q[x]$ for $0 \leq r \leq \infty$ is defined to be the subalgebra of $Z_{(p)}$ generated by $1, x, (p!)^{-1}x^p, \dots, (p^r!)^{-1}x^{p^r}$. Analogously $\Delta(\xi, r) \subset Q[\xi]$ for $0 \leq r \leq \infty$ is defined to be the subalgebra over $Z_{(p)}$ generated by

$$1, \xi, (p!)^{-1}\xi(\xi-1) \cdots (\xi-p+1), \dots, (p^r!)^{-1}\xi(\xi-1) \cdots (\xi-p^r+1).$$

We consider the zero dimensional complex K -homology group with coefficients $Z_{(p)}$, the integers localized at the prime ideal (p) .

PROPOSITION 2.1. (a) $H_*(X, Z_{(p)}) \cong \Gamma(x_2, \infty) \otimes Z_{(p)}[x_4, x_6, \dots, x_{2n}, \dots]$ as rings, where both factors are sub-Hopf algebras.

(b) $K_0(X, Z_{(p)}) \cong \Delta(\xi_2, \infty) \otimes Z_{(p)}[\xi_4, \xi_6, \dots, \xi_{2n}, \dots]$ as rings, where both factors are sub-Hopf algebras.

PROOF. (a) Let $A = H_*(X, Z_{(p)})$. Then by Theorem 1.1 of [3] we may write $A = \otimes E(2r)$ as a Hopf algebra where r runs through those positive integers with $(r, p) = 1$ and as an algebra $E(2r)$ is generated by classes with dimensions $2p^i r$, $i \geq 0$. Therefore $A^* = \otimes E(2r)^* = H^*(X, Z_{(p)})$. Now $X_p = BU_p$ and so the conditions of Theorem 3.1(c) of [4] are satisfied for $r > 1$ and we deduce that $E(2r)^* \cong C(2r, 0)$ in the notation of that paper. But $C(2r, 0) \cong C(2r, 0)^*$ as a Hopf algebra and as an algebra

$$C(2r, 0) \cong Z_{(p)}[x_{2r}, x_{2p}, \dots, x_{2p^i r}, \dots].$$

It remains to consider $E(2)$. Theorem 1.1 of [5] implies that $E(2) \cong \Gamma(x_2, r) \otimes D$ where D is a polynomial algebra. The condition on the Frobenius map implies that $r \geq 1$ and so $\dim Q^{2p}\{E(2) \otimes Z/pZ\} = 2$. Thus $\dim P^{2p}\{(E(2) \otimes Z/pZ)^*\} = 2$ and as $(E(2) \otimes Z/pZ)^*$ is a polynomial algebra it follows that

$$\dim P^{2i}\{(E(2) \otimes Z/pZ)^*\} \geq 2 \quad \text{for } i = p^\alpha, \alpha \geq 1.$$

Therefore $\dim Q^{2i}\{E(2) \otimes Z/pZ\} \geq 2$ for $i = p^\alpha$, $\alpha \geq 1$. It follows that $E(2) \cong \Gamma(x_2, \infty) \otimes D$ and so

$$H_*(X, Z_{(p)}) \cong \Gamma(x_2, \infty) \otimes Z_{(p)}[x_4, x_6, \dots, x_{2n}, \dots]$$

as rings.

Clearly $\Gamma(x_2, \infty)$ is a sub-Hopf algebra. By modifying the generators x_{2n} inductively using the basic technique of [3] and in particular of Lemma 2.2 and its proof, one can ensure that the polynomial algebra is a sub-Hopf algebra.

(b) Let $h: X \rightarrow K(Z_{(p)}, 2)$ realize a generator of $H^2(X, Z_{(p)})$. Then h is an H -map and by part (a),

$$h_*: H_*(X, Z_{(p)}) \rightarrow H_*(K(Z_{(p)}, 2), Z_{(p)}) \cong \Gamma(x'_2, \infty)$$

is surjective. Therefore

$$h_0: K_0(X, Z_{(p)}) \rightarrow K_0(K(Z_{(p)}, 2), Z_{(p)}) \cong \Delta(\xi'_2, \infty)$$

is surjective and so is

$$h_0: K_0(X, Z_{(p)})_{2n} \rightarrow K_0(K(Z_{(p)}, 2), Z_{(p)})_{2n}$$

in the CW-filtration for each n . Now by again modifying the generators x_{2i} for $i > 1$ given in part (a), we can ensure that $h_*(x_{2i}) = 0$ for $i > 1$. We choose $\xi_{2i} \in K_0(X, Z_{(p)})_{2i}$ whose image in

$$K_0(X, Z_{(p)})_{2i}/K_0(X, Z_{(p)})_{2i-2} \cong H_{2i}(X, Z_{(p)})$$

is x_{2i} and so again by modifying the original choices of the ξ_{2i} we can ensure that $h_0(\xi_{2i}) = 0$ for $i > 1$. Therefore for each i ,

$$(p^i!)^{-1} \xi_2(\xi_2 - 1) \cdots (\xi_2 - p^i + 1) \in K_0(X, Z_{(p)})$$

and it follows that $K_0(X, Z_{(p)}) \cong \Delta(\xi_2, \infty) \otimes Z_{(p)}[\xi_4, \xi_6, \dots, \xi_{2n}, \dots]$ as rings.

Again it is clear that $\Delta(\xi_2, \infty)$ is a sub-Hopf algebra and by modifying the original ξ_{2i} for $i > 1$ as in the similar step in (a) we can ensure that the polynomial algebra is a sub-Hopf algebra. This completes the proof.

It follows from Theorem 3.1 of [4] that the Hopf algebra structure of $H_*(X, Z_{(p)})$ is well defined. In fact $H_*(X, Z_{(p)}) \cong \Gamma(x_2, \infty) \otimes C(2, 1) \otimes \{ \otimes C(2r, 0) \}$ where $r > 1$ and $(r, p) = 1$ as Hopf algebras. In particular $H_*(X, Z_{(p)}) \cong H_*(BU_p^\otimes, Z_{(p)})$.

We can now complete the proofs. We recall [2, 6] that as an H -space

$$BU_p^\otimes = K(Z_{(p)}, 2) \times BSU_p^1 \times BU_p^2 \times \cdots \times BU_p^{p-1}$$

and that H -maps $g_i: X \rightarrow BU_p^i$ for $i > 1$ were constructed in §4 of [4] inducing monomorphisms of Hopf algebras

$$g_i^*: H^*(BU_p^i, Z/pZ) \rightarrow H^*(X, Z/pZ).$$

Therefore we must construct an H -map $g_1: X \rightarrow K(Z_{(p)}, 2) \times BSU_p^1$ with similar properties, for then $g_1 \times g_2 \times \cdots \times g_{p-1}: X \rightarrow BU_p^\otimes$ will be an H -equivalence.

The standard decomposition of $BU_p^\otimes$ implies that $K_0(X, Z_{(p)})$ and $K^0(X, Z_{(p)})$ decompose canonically into $(p-1)$ -summands and we assume that the generators ξ_{2i} of Proposition 2.1(b) lie in their appropriate summands as in [4]. Now $QK_0(X, Z_{(p)}) \cong Q \oplus \{ \oplus (Z_{(p)})^\infty \}$ where the $Z_{(p)}$ summands are generated by the images of the ξ_{2i} . (The Q on the right-hand side denotes the rationals [1].) Therefore

$$PK^0(X, Z_{(p)}) \cong \text{Hom}(QK_0(X, Z_{(p)}), Z_{(p)}) \cong \Pi(Z_{(p)})^\infty$$

generated by η_{2i} for $i > 1$ such that $\eta_{2i}(\xi_{2j}) = \delta_{i,j}$ and can be taken to lie in their appropriate summands. We consider $\eta_{2p} \in K^0(X, Z_{(p)})$. As it is primitive, it represents an H -map $\eta_{2p}: X \rightarrow BU_p^{1,\oplus}$; it has exact CW-filtration $2p$ and the element in the associated graded group $H^{2p}(X, Z_{(p)})$ is a polynomial generator y_{2p} . Now since $BU_p^1 = K(Z_{(p)}, 2) \times BSU_p^1$ as a space and $\eta_{2p} \in K^0(X, Z_{(p)})_{2p}$, η_{2p} induces $\eta'_{2p}: X \rightarrow BSU_p^1$ with $i\eta'_{2p} = \eta_{2p}$ where $i: BSU_p^1 \rightarrow BU_p^1$ is the inclusion. As i is an H -map, η'_{2p} is an H -map. Further since $(\eta'_{2p})!: [BSU_p^1, BU] \rightarrow [X, BU]$ maps i to η_{2p} and BSU_p^1 is $(2p-1)$ -connected, the induced homomorphism

$$(\eta'_{2p})^*: QH^{2p}(BSU_p^1, Z_{(p)}) \rightarrow QH^{2p}(X, Z_{(p)})$$

is an isomorphism.

Let $h: X \rightarrow K(Z_{(p)}, 2)$ realize a generator of $H^2(X, Z_{(p)})$. Then $g_1 = h \times \eta'_{2p}: X \rightarrow K(Z_{(p)}, 2) \times BSU_p^1$ is an H -map and induces

$$g_1^*: QH^{2i}(K(Z_{(p)}, 2) \times BSU_p^1, Z_{(p)}) \rightarrow QH^{2i}(X, Z_{(p)})$$

which is an isomorphism for $i = 1$ and p . Therefore as in the proof of Theorem 4.4 of [4], it induces an isomorphism for all i of the form $1 + k(p - 1)$. Hence $k = g_1 \times g_2 \times \cdots \times g_p: X \rightarrow BU_p^\otimes$ is the mod p H -equivalence we seek.

3. The proof of Theorem 1.4. Let $X \simeq_2 BSp$ where X is a standard H -space. We must show that $X \simeq_2 BSp^\otimes$ as an H -space and to do this it is sufficient to construct an H -map $k: X_2 \rightarrow BSp_2^\otimes$ which induces an isomorphism of cohomology groups in dimension 4.

LEMMA 3.1. *Let $k: X_2 \rightarrow X_2$ induce an isomorphism $k^*: H^4(X_2, Z/2Z) \rightarrow H^4(X_2, Z/2Z)$. Then k is a homotopy equivalence.*

PROOF. If we can show that $k^*: QH^{4i}(X_2, Z/2Z) \rightarrow QH^{4i}(X_2, Z/2Z)$ is an isomorphism for $i = 2$, then the action of the Steenrod algebra in $H^*(BSp, Z/2Z)$ ensures that k^* is an isomorphism for $i > 2$. Thus $k^*: H^*(X_2, Z/2Z) \rightarrow H^*(X_2, Z/2Z)$ is an isomorphism. The lemma then follows from Whitehead's theorem.

The result needed in dimension 8 is most easily proved using complex K -theory. We know that

$$K^0(X_2, Z_{(2)}) = Z_{(2)}[[\eta_4, \eta_8, \dots, \eta_{4n}, \dots]]$$

where η_{4n} has exact CW-filtration $4n$ and by direct and standard calculations we can choose η_4 and η_8 such that $\psi^2(\eta_4) = 4\eta_4 + 2\eta_8 + \eta_4^2$. We consider $k!: QK^0(X_2, Z_{(2)}) \rightarrow QK^0(X_2, Z_{(2)})$ and work mod $QK(X_2, Z_{(2)})_9$. Then $k!(\eta_4) = \lambda\eta_4 + \mu\eta_8$ and $k!(\eta_8) = \nu\eta_8$. The hypotheses imply that λ is a unit in $Z_{(2)}$ and we wish to prove the same for ν . Since $\psi^2(\eta_4) = 4\eta_4 + 2\eta_8$ and $\psi^2(\eta_8) = 16\eta_8$, from $\psi^2(k!\eta_4) = k!\psi^2(\eta_4)$ we deduce that $2\lambda = 2\nu \bmod 4$ and so ν is a unit. This completes the proof of the lemma.

Theorem 3.1 and the comments which follow Corollary 3.2 in [4] imply that the Hopf algebra structure of $H_*(X_2, Z_{(2)})$ is well defined. In the notation used there $H_*(X_2, Z_{(2)}) \cong \otimes C(2r, 1)$, where r runs over all positive odd integers. In particular,

$$H_*(X_2, Z_{(2)}) \cong Z_{(2)}[x_4, x_8, \dots, x_{4n}, \dots]$$

and so $K_0(X_2, Z_{(2)}) \cong Z_{(2)}[\xi_4, \xi_8, \dots, \xi_{4n}, \dots]$ and $QK_0(X_2, Z_{(2)}) \cong \oplus (Z_{(2)})^\infty$ where the $Z_{(2)}$ summands are generated by the images of the ξ_{4i} . Therefore

$$PK^0(X_2, Z_{(2)}) = \text{Hom}(QK_0(X_2, Z_{(2)}), Z_{(2)}) \cong \Pi(Z_{(2)})^\infty$$

generated by $\{\eta_{4i}\}$ where $\eta_{4i}(\xi_{4j}) = \delta_{i,j}$. Then $\eta_4: X_2 \rightarrow BU_2^\otimes$ is an H -map and regarded as an element of $K^0(X_2, Z_{(2)})_4$ has image y_4 a generator in $H^4(X_2, Z_{(2)})$. As $BU_2 = K(Z_{(2)}, 2) \times BSU_2$, $\eta_4^*: H^4(BU_2, Z_{(2)}) \rightarrow H^4(X_2, Z_{(2)})$ is an isomorphism. We must show that $\eta_4: X_2 \rightarrow BU_2$ lifts to an H -map $\eta'_4: X_2 \rightarrow BSp_2$ inducing an isomorphism of 4 dimensional cohomology groups.

We write $KO(\)$ and $KH(\)$ for real and symplectic zero dimensional cohomology groups with $Z_{(2)}$ -coefficients. Now X_2 and $X_2 \times X_2$ have local cellular structures in which all cells have dimension $4n$. Thus their real and symplectic K -groups

are torsion free and the complexification and restriction homomorphisms $c: KO(\) \rightarrow K^0(\)$ and $c': KH(\) \rightarrow K^0(\)$ are monomorphisms. We consider $KO(\)$ and $KH(\)$ as subgroups of $K^0(\)$. We will require that for X_2 and $X_2 \times X_2$

$$(3.1) \quad K^0(\) = KO(\) + KH(\) \quad \text{and} \quad 2K^0(\) = KO(\) \cap KH(\).$$

For finite complexes with cells with dimensions $4n$, (3.1) can be proved by induction since $c: KO(S^{4k}, Z_{(2)}) \rightarrow K^0(S^{4k}, Z_{(2)})$ is an isomorphism if k is even and has cokernel $Z/2Z$ if k is odd and $c': KH(S^{4k}, Z_{(2)}) \rightarrow K^0(S^{4k}, Z_{(2)})$ is an isomorphism if k is odd and has cokernel $Z/2Z$ if k is even. The conclusion for X_2 and $X_2 \times X_2$ is obtained by taking inverse limits over the finite skeleta.

Now using (3.1) we write $\eta_4 = r + q$ where $r \in \widetilde{KO}(X, Z_{(2)})$ and $q \in \widetilde{KH}(X, Z_{(2)})$. We consider $\tilde{m}!: K^0(X, Z_{(2)}) \rightarrow K^0(X \times X, Z_{(2)})$ where $\tilde{m}! = m! - \pi_1! - \pi_2!$ so that $\tilde{m}!(\eta_4) = 0$. Thus $\zeta = \tilde{m}!(q) = -\tilde{m}!(r)$ is both real and symplectic and therefore by (3.1) $\zeta = 0 \bmod 2$. Now the Hopf algebra structure of $H^*(X_2, Z_{(2)}) \cong \otimes C(2r, 1)$ is well understood and $PH^*(X_2, Z_{(2)}) \cong \oplus (Z_{(2)})^\infty$ generated by y_{4i} elements in the associated graded group corresponding to η_{4i} . In $H^*(X_2, Z_{(2)})$ if a homogeneous element z is primitive mod 2, then $z = z' + 2z''$ where z' is primitive. By working inductively along the filtration in $K^0(X_2, Z_{(2)})$ one deduces that if $\tilde{m}!(\phi) = 0 \bmod 2$, then $\phi = \phi' + 2\phi''$ where $\tilde{m}!(\phi') = 0$. We deduce that $q = q' + 2q''$ where $q' \in K^0(X_2, Z_{(2)})$ is primitive and as $q' = q - 2q''$ it lies in $KH(X_2, Z_{(2)})$. Let $r' = r + 2q''$. Then $\eta_4 = q' + r'$ where $q': X_2 \rightarrow BSp_2$ is an H -map. Now $r' \in K^0(X_2, Z_{(2)})_4$ has an image in $H^4(X_2, Z_{(2)})$ which is divisible by 2, since $KO(S^4) \rightarrow K^0(S^4)$ has cokernel $Z/2Z$. Thus the image of q' in $H^4(X_2, Z_{(2)})$ is a generator.

We set $k = q': X_2 \rightarrow BSp_2$ which has all the properties required of it to complete the proof.

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