GENERALIZED INTERSECTION MULTIPLICITIES OF MODULES. II

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ABSTRACT. The vanishing conjecture for intersection multiplicities in dimensions ≤ 5 is true in greater generality than previously known.

In this note we prove a more general theorem than Corollary 2.5 of [D1]. Since both the vanishing and nonvanishing conjectures are not true in general [DHM], it is now important to understand the following question: What can we say about

$$\chi^R(M, N) = \sum_{i=0}^d (-1)^i l(\text{Tor}_i^R(M, N)),$$

where R is a Gorenstein ring, and M and N are modules over R with finite projective dimension such that $l(M \otimes_R N) < \infty$, $\dim M + \dim N \leq \dim R$, and $d = \text{p.d. } M$?

The following theorems partially answer the question up to dimension 5.

Throughout §1 all tensor products, Tor's and Euler characteristics are computed over R .

1.1. THEOREM. *Let R be a Gorenstein ring of dimension ≤ 5 . Let M and N be modules of finite projective dimension with $l(M \otimes N) < \infty$ and $\dim M + \dim N < \dim R$. Then $\chi(M, N) = 0$.*

PROOF. First we note that §§1.1 and 1.3 of [D1] hold true in the above case. The sufficient part of Theorem 2.4 of [D1] is still valid, though the statement has to be changed a bit in the following way:

1.2. THEOREM. *Let R be a Gorenstein ring of dim N . If for any two perfect modules M and N with (i) $\dim M + \dim N = \dim R$, and (ii) $l(M \otimes N) < \infty$, $l(M \otimes N) = l(M \otimes \check{N})$, then the vanishing conjecture holds in R for every pair of modules of finite projective dimension. Here $\check{N} = \text{Ext}^{n-r}(N, R)$, $r = \dim N$.*

Proof of this theorem can be obtained by following the argument of (2.4) of [D1].

To prove our theorem, by 1.2 it is enough to prove that

$$(1) \quad l(T \otimes_R Q) = l(T \otimes_R \check{Q}), \dots,$$

where T and Q are any two perfect modules over R with $l(T \otimes Q) < \infty$ and $\dim T + \dim Q = \dim R$.

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Case 1. $\dim T = 5$, i.e., T is free, hence $l(Q) < \infty$. Since $l(Q) = l(\check{Q})$, $l(T \otimes Q) = l(T \otimes \check{Q})$.

Case 2. $\dim T = 4$. In this case by killing an R -sequence which is also an M -sequence of length four contained in $\text{Ann}_R Q$, we reduce the proof to proving (1) on a Gorenstein ring of dimension 1 and, hence, we are done.

Case 3. $\dim T = 3$. In this case killing an M -sequence which is also an R -sequence of length 3, contained in $\text{Ann}_R Q$, we reduce the proof to proving (1) on a Gorenstein ring of dim 2 and we are done by (1.1) of [D1].

Similar arguments hold true when $\dim R \leq 4$.

The next theorem throws a little more light on the nonvanishing conjecture. (This is a changed version of Proposition (3.4) of [D1].)

1.3. THEOREM. *Let R be a complete Gorenstein ring of dim 5 whose coefficient ring is a discrete valuation ring V , and let p be a generator of the maximal ideal of V . Let M and N be two modules over R such that (i) M is perfect and $\text{p.d. } N < \infty$, (ii) $l(M \otimes_R N) < \infty$, (iii) $\dim M + \dim N = 5$, (iv) $pN = 0$, and p is a n.z.d. on M . Then $\chi(M, N) > 0$.*

For the proof we refer the reader to (1.8) and Corollaries 5 and 6 of [D2].

REMARK. Theorems (2.2) and (2.4) of [D1] and Theorem 1.1 can be proved with the assumption that R is a Cohen-Macaulay ring. The techniques are the same; one has only to consider spectral sequences $\text{Ext}_S^i(\text{Tor}_j^R(M, N), S)$ and $\text{Ext}_R^i(M, \text{Ext}_S^j(N, S))$, and instead of taking $\check{M}, \check{N}$ as defined in the Gorenstein case with $\text{p.d. } M = d$, say, one should consider $\check{M} = \text{Ext}_R^d(M, R)$ and $\check{N} = \text{Ext}_S^{n-r}(N, S)$, $r = \dim N$. Here S is a Gorenstein ring such that $R = S/I$ and $\dim R = \dim S$ (assuming R to be complete).

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