ABELIAN AND TAUBERIAN THEOREMS FOR 0-REGULARLY VARYING FUNCTIONS

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ABSTRACT. A general kernel Abelian and Tauberian theorem is proved for functions f satisfying $\lim_{t \rightarrow \infty} f(tx)/f(t) < \infty$ for $x > 0$.

Introduction. In this paper we prove Abelian and Tauberian theorems for integral transforms for two classes of functions. First we give the definition of the classes of functions we consider.

DEFINITION (SEE [1]). A measurable function $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is *0-regularly varying* (notation $f \in \text{RO}$) if one of the following equivalent conditions holds:

(i)

$$(1) \quad \Phi_f(x) := \overline{\lim}_{t \rightarrow \infty} \frac{f(tx)}{f(t)} < \infty \quad \text{for } x > 0.$$

(ii) There exist constants $c_1, c_2, t_0 \geq 0$ and $\alpha, \beta \in \mathbf{R}$ such that for $t > t_0$ and $x \geq 1$

$$(2) \quad c_1 x^\alpha \leq \frac{f(tx)}{f(t)} \leq c_2 x^\beta.$$

(iii)

$$(3) \quad \lim_{x \rightarrow \infty} \frac{\log \overline{\lim}_{t \rightarrow \infty} f(tx)/f(t)}{\log x} < \infty$$

and

$$(4) \quad \lim_{x \rightarrow \infty} \frac{\log \underline{\lim}_{t \rightarrow \infty} f(tx)/f(t)}{\log x} > -\infty.$$

(iv) There exists a positive nondecreasing function ρ , a positive nonincreasing function ψ and constants $\alpha, \beta \in \mathbf{R}$ such that

$$(5) \quad f(t) \asymp t^\alpha \rho(t) \asymp t^\beta \psi(t) \quad (t \rightarrow \infty).$$

(We use the symbol $a(t) \asymp b(t)$ ($t \rightarrow \infty$) to mean $a(t), b(t) > 0$ and $\log a(t)/b(t)$ is bounded for $t > \infty$.)

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The left-hand side of (3) is called the upper index of f ($\overline{\text{index}} f$) and the left-hand side of (4) is called the lower index of f ($\underline{\text{index}} f$). If $f \in \text{RO}$ we can take any $\alpha < \underline{\text{index}} f$ and $\beta > \overline{\text{index}} f$ in (2) and (5). For a review of properties of 0-regularly varying functions the reader is referred to [2].

DEFINITION. A nondecreasing function $f: \mathbf{R}^+ \rightarrow \mathbf{R}$ is *asymptotically balanced* if there exists a constant $x > 1$ and a function a such that

$$(6) \quad f(tx) - f(t) \asymp a(t) \quad (t \rightarrow \infty).$$

It is well known that (6) implies $a \in \text{RO}$ (see [5]).

We study integral transforms of the above classes of functions and use the following notation:

$$(7) \quad \hat{f}(\sigma) = -\sigma \int_0^\infty \rho'(s) f(s) ds, \quad \sigma > 0.$$

In this paper the function $\rho: [0, \infty) \rightarrow [0, \infty)$ is differentiable with $\rho'(x) \leq 0$ for $x > 0$, $\rho(\infty) = 0$ and $\rho(x) > 0$ for some $x > 0$. The integral in (7) is supposed to be finite for $\sigma > 0$.

In case the limit in (1) exists it is possible to deduce a Tauberian theorem using Wiener's approach. This is shown in a paper by Bingham and Teugels [3]. A Tauberian theorem concerning functions f satisfying (6) with $\asymp$ replaced by $\sim$, where the function a is slowly varying, is proved in [4]. For the case of the Laplace-Stieltjes transform a Tauberian theorem for nondecreasing 0-regularly varying functions is given in de Haan and Stadtmüller [6].

RESULTS. In the following, c_i ($i = 1, 2, \dots$) denotes a positive finite constant not necessarily the same at each appearance.

First we give the following Abelian result.

THEOREM 1. Suppose $f \in \text{RO}$ is locally bounded, $\alpha := \underline{\text{index}} f$, $\beta := \overline{\text{index}} f$,

$$(8) \quad -\int_0^1 s^{\alpha-\delta} \rho'(s) ds < \infty, \quad -\int_1^\infty s^{\beta+\delta} \rho'(s) ds < \infty$$

for some $\delta > 0$.

The above assumptions imply

$$(9) \quad f(t) \asymp \hat{f}(1/t) \quad (t \rightarrow \infty),$$

hence

$$(10) \quad \lim_{t \rightarrow \infty} \frac{\hat{f}(1/tx)}{\hat{f}(1/t)} < \infty \quad \text{for } x > 0.$$

In order to prove this result we need the following.

LEMMA. Suppose $0 < \underline{\text{index}} f \leq \overline{\text{index}} f < \infty$ and f is locally bounded on $\mathbf{R}^+$. Then there exist t_0 and c_1 such that for $t \geq t_0$ and $0 < x \leq 1$

$$(11) \quad f(tx)/f(t) \leq c_1.$$

Suppose $\alpha := \underline{\text{index}} f \leq 0$, $\overline{\text{index}} f < \infty$ and f is locally bounded on $\mathbf{R}^+$. For $\delta > 0$ arbitrary, there exist t_1 and c_2 such that for $t \geq t_1$ and $0 < x \leq 1$

$$(12) \quad f(tx)/f(t) \leq c_2 x^{\alpha-\delta}.$$

PROOF. The proof of this Lemma is given by application of the first inequality in (2) and applying the first result to the function $t^{-\alpha+\delta}f(t)$.

PROOF OF THEOREM 1. Take $\delta > 0$ such that (8) holds true. Then using (11) and (2) we find

$$(13) \quad \frac{\hat{f}(1/t)}{f(t)} = -\int_0^1 \rho'(s) \frac{(ts)^{-\alpha+\delta} f(ts)}{t^{-\alpha+\delta} f(t)} s^{\alpha-\delta} ds - \int_1^\infty \rho'(s) \frac{f(ts)}{f(t)} ds \\ \leq -c_1 \int_0^1 \rho'(s) s^{\alpha-\delta} ds - c_2 \int_1^\infty \rho'(s) s^{\beta+\delta} ds.$$

Take a nondecreasing function ϕ on (t_0, ∞) such that $f(t) \asymp t^{\alpha-\delta} \phi(t)$ for $t > t_0$. Then for $t > t_0$

$$(14) \quad \hat{f}\left(\frac{1}{t}\right) \geq -\int_1^\infty \rho'(s) f(ts) ds \geq -c_3 t^{\alpha-\delta} \int_1^\infty \rho'(s) s^{\alpha-\delta} \phi(st) ds \\ \geq -c_3 t^{\alpha-\delta} \phi(t) \int_1^\infty \rho'(s) s^{\alpha-\delta} ds \geq c_4 f(t).$$

Now (1), (13) and (14) imply (9) and (10).

Next we prove a Tauberian converse of Theorem 1. Theorem 1 concerns functions f tending to infinity as well as functions tending to zero. The following two theorems give converse statements for functions f bounded away from zero (Theorem 2) and infinity (Theorem 3).

THEOREM 2. Suppose $f: [0, \infty) \rightarrow [0, \infty)$ is of bounded variation on finite intervals, and satisfies

$$(15) \quad f(0+) = 0,$$

$$(16) \quad \lim_{x \rightarrow \infty} f(x) > 0 \quad \text{and} \quad \overline{\lim}_{x \rightarrow \infty} f(x) = \infty,$$

and, for $\lambda > 1$ fixed,

$$(17) \quad \lim_{t \rightarrow \infty} \inf_{t \leq t' \leq \lambda t} \{f(t') - f(t)\} > -\infty.$$

Suppose ρ satisfies

$$(18) \quad \sum_{k=1}^{\infty} k \{ \rho(\lambda^{-k-1}) - \rho(\lambda^{-k}) \} < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \rho(\lambda^k) < \infty.$$

Let

$$(19) \quad \Phi_f(x) := \overline{\lim}_{t \rightarrow \infty} \frac{\hat{f}(1/tx)}{\hat{f}(1/t)} < \infty \quad \text{for } x > 1.$$

Then $a = \overline{\text{index}} \hat{f}(1/t)$ exists.

Suppose

$$(20) \quad -\int_1^\infty t^\alpha \rho'(t) dt < \infty \quad \text{for some } \alpha > a.$$

Then the above assumptions imply (9), and hence,

$$(21) \quad \lim_{t \rightarrow \infty} \frac{f(tx)}{f(t)} < \infty \quad \text{for } x > 1.$$

PROOF. Without loss of generality we may suppose $\rho(1) > 0$. For $T = 1/\sigma > 0$ we have

$$\hat{f}(\sigma) = \rho(1)f(T) - \int_0^T \sigma \rho'(\sigma t) f(t) dt + \int_T^\infty \rho(\sigma t) df(t).$$

Using (17) we have

$$\begin{aligned} \int_T^\infty \rho(\sigma t) df(t) &= \sum_{k=0}^\infty \int_{T\lambda^k}^{T\lambda^{k+1}} \rho(\sigma t) df(t) \\ &\geq \sum_{k=0}^\infty \rho(\lambda^k) \inf\{f(\xi) - f(T\lambda^k)\} \\ &\geq -c \sum_{k=0}^\infty \rho(\lambda^k) := -c_1, \end{aligned}$$

where the infimum is taken over $\xi \in [T\lambda^k, T\lambda^{k+1}]$. This proves

$$(22) \quad \hat{f}(\sigma) \geq \rho(1)f(1/\sigma) - c_1 \quad (\sigma > 0).$$

For $\sigma > 0$ we write

$$(23) \quad \hat{f}(\sigma) = -\int_0^\beta f(tT)\rho'(t) dt - \int_\beta^\infty f(tT)\rho'(t) dt,$$

where $\beta > 0$ is specified later.

Using (17) for $t > 0$ there exists $t_0 \in [t, \lambda t]$ such that

$$(24) \quad \begin{aligned} \inf_{t \leq t' \leq \lambda t} \{f(\lambda t) - f(t')\} &\geq f(\lambda t) - f(t_0) - 1 \\ &\geq \inf_{t_0 \leq t' \leq \lambda t_0} \{f(t') - f(t_0)\} - 1 \geq -c \end{aligned}$$

since $t \leq t_0 \leq \lambda t \leq \lambda t_0$. Since ρ is nonincreasing we have

$$\begin{aligned} -\int_0^\beta f(tT)\rho'(t) dt &= -\sum_{k=0}^\infty \int_{\beta\lambda^{-k-1}}^{\beta\lambda^{-k}} f(tT)\rho'(t) dt \\ &\leq -\sum_{k=0}^\infty \sup_{\beta\lambda^{-k-1}T \leq \xi \leq \beta\lambda^{-k}T} f(\xi) \{\rho(\beta\lambda^{-k}) - \rho(\beta\lambda^{-k-1})\} \\ &\leq \sum_{k=0}^\infty [f(\beta\lambda^{-k}T) + c] \{\rho(\beta\lambda^{-k-1}) - \rho(\beta\lambda^{-k})\}, \end{aligned}$$

the last inequality being justified by (24).

Repeated application of (17) gives, for $k = 1, 2, \dots$,

$$f(\beta\lambda^{-k}T) \leq f(\beta T) + kc.$$

Hence

$$(25) \quad -\int_0^\beta f(tT)\rho'(t) dt \leq c_2 f(\beta T) + c_3,$$

where $c_2 = \rho(0) - \rho(\beta)$ and $c_3 = c \sum_{k=0}^{\infty} (k+1) \{ \rho(\beta \lambda^{-k-1}) - \rho(\beta \lambda^{-k}) \}$. In order to estimate the second integral in (23) we note that by (2) there exist constants c_4 and σ_0 such that for $t \geq 1$ and $\sigma < \sigma_0$

$$\hat{f}(\sigma/t) \leq c_4 t^\alpha \hat{f}(\sigma).$$

Using the above inequality and (22) we have for $\sigma < \sigma_0$ and $\beta \geq 1$

$$(26) \quad - \int_{\beta}^{\infty} f(tT) \rho'(t) dt \leq -\rho(1)^{-1} \int_{\beta}^{\infty} \left\{ \hat{f}\left(\frac{\sigma}{t}\right) + c_1 \right\} \rho'(t) dt \\ \leq c_5 \hat{f}(\sigma) + c_6,$$

where $c_5 = -c_4 \rho(1)^{-1} \int_{\beta}^{\infty} t^\alpha \rho'(t) dt < \infty$ and $c_6 = c_1 \rho(1)^{-1} \rho(\beta)$. Combination of (23), (25) and (26) gives for $\sigma < \sigma_0$ and $\beta \geq 1$

$$\hat{f}(\sigma) \leq c_2 f(\beta/\sigma) + c_3 + c_5 \hat{f}(\sigma) + c_6.$$

Next we take $\beta \geq 1$ such that $\rho(\beta) < \rho(0)$ and $c_5 \leq \frac{1}{2}$. Then for $0 < \sigma < \sigma_0$

$$(27) \quad \hat{f}(\sigma) \leq 2c_2 f(\beta/\sigma) + 2(c_3 + c_6).$$

Then (19), (22) and (27) imply (9) if f is unbounded (i.e. $\hat{f}$ is unbounded), hence (21).

REMARK. If f is bounded, Theorem 1 can be applied (suppose ρ satisfies (8) with $\alpha = 0$).

Karamata [7] considered the case $\hat{f}(1/t) = O(g(t))$ ($t \rightarrow \infty$) with $g \in \text{RO}$.

DEFINITION (SEE [1]). A function f is said to be almost decreasing on $\mathbf{R}^+$ if there exists a constant $0 < m \leq 1$ such that

$$(28) \quad f(x) \geq mf(y) \quad \text{for each } y \geq x > 0.$$

THEOREM 3. Suppose $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is almost decreasing on $\mathbf{R}^+$. Moreover, suppose $g \in \text{RO}$ with $a := \underline{\text{index}} g \leq 0$, where the function g is defined by $g(t) = \hat{f}(1/t)$ ($t > 0$). Suppose

$$(29) \quad - \int_0^1 t^\alpha \rho'(t) dt < \infty \quad \text{for some } \alpha < a.$$

Then the above assumptions imply (9), hence $\underline{\text{index}} f \leq 0$.

PROOF. Analogous to the proof of Theorem 1 in de Haan and Stadtmüller [6].

Take $b > 0$ such that $\rho(b) < \rho(0)$. Then since f is almost decreasing, for $t > 0$,

$$(30) \quad \hat{f}\left(\frac{1}{t}\right) \geq - \int_0^b \rho'(s) f(ts) ds \geq +c_1 f(tb) \{ \rho(0) - \rho(b) \} = c_2 f(tb).$$

Conversely, using (30) we find

$$(31) \quad \hat{f}(\sigma) = - \int_0^\beta \rho'(x) f\left(\frac{x}{\sigma}\right) dx - \int_\beta^\infty \rho'(x) f\left(\frac{x}{\sigma}\right) dx \\ \leq c_3 f\left(\frac{\beta}{\sigma}\right) \rho(\beta) - c_4 \int_0^\beta \rho'(x) \hat{f}\left(\frac{b\sigma}{x}\right) dx,$$

where $\beta > 0$ is specified later.

Application of the second part of the Lemma gives for $\sigma \leq \sigma_0$

$$- \int_0^\beta \rho'(x) \hat{f}\left(\frac{b\sigma}{x}\right) dx \leq -c_5 \hat{f}(\sigma) \int_0^\beta \rho'(x) x^\alpha dx.$$

Take $0 < \beta < b$ such that

$$(32) \quad \rho(\beta) > 0 \quad \text{and} \quad -c_4 c_5 \int_0^\beta \rho'(x) x^\alpha dx < \frac{1}{2}.$$

Then (31) and (32) imply

$$(33) \quad \hat{f}(\sigma) \leq 2c_3 \rho(\beta) f(\beta/\sigma) \quad (\sigma \leq \sigma_0).$$

Now (30), (33) and $g \in \text{RO}$ imply $\hat{f}(1/t) \asymp f(t) \ (t \rightarrow \infty)$.

Next we prove a theorem concerning asymptotically balanced functions.

THEOREM 4. Suppose $U: \mathbf{R}^+ \rightarrow \mathbf{R}$ and there exist a constant $x > 1$ and a positive function f such that

$$(34) \quad U(tx) - U(t) \asymp f(t) \quad (t \rightarrow \infty)$$

with $f \in \text{RO}$ locally bounded, $\alpha := \underline{\text{index}} f$ and $\beta := \overline{\text{index}} f$. Suppose $\hat{U}(t)$ exists for $t > 0$ and (8) is satisfied. Then the above assumptions imply

$$(35) \quad \hat{U}(1/tx) - \hat{U}(1/t) \asymp f(t) \quad (t \rightarrow \infty).$$

The proof of the above theorem is a consequence of Theorem 1. In order to prove a converse statement we suppose the function U to be nondecreasing.

THEOREM 5. Suppose $U: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is nondecreasing, $U(0+) = 0$ and define the function f by

$$(36) \quad f(t) = t^{-1} \int_0^t s dU(s).$$

Suppose for some $x > 1$

$$(37) \quad \hat{U}(1/tx) - \hat{U}(1/t) \asymp a(t) \quad (t \rightarrow \infty)$$

where $\underline{\text{index}} a > -1$. If one of the following assumptions is satisfied:

(a) (16), (17), (18) and

$$-\int_1^\infty t^\alpha \rho'(t) dt < \infty \quad \text{for some } \alpha > \underline{\text{index}} a,$$

(b) f is almost decreasing on $\mathbf{R}^+$, $\underline{\text{index}} a \leq 0$ and

$$-\int_0^1 t^\alpha \rho'(t) dt < \infty \quad \text{for some } \alpha < \underline{\text{index}} a,$$

then

$$(38) \quad U(tx) - U(t) \asymp a(t) \quad (t \rightarrow \infty).$$

PROOF. Note that

$$\hat{U}\left(\frac{1}{x}\right) - \frac{1}{x} \int_0^x \hat{U}\left(\frac{1}{s}\right) ds = - \int_0^\infty \rho'(s) f(sx) ds = \hat{f}\left(\frac{1}{x}\right) \quad \text{for } x > 0.$$

By de Haan and Stadtmüller [6], (37) is equivalent to

$$t^{-1} \int_0^t s d\hat{U}(1/s) \asymp a(t) \quad (t \rightarrow \infty)$$

with $a \in \text{RO}$, and

$$\lim_{t \rightarrow \infty} a(tx)/a(t) > x^{-1} \quad \text{for some } x > 1.$$

The proof is finished by application of Theorems 2 and 3.

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