SOME TOPOLOGICAL PROPERTIES OF THE 1-SET-CONTRACTIONS

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ABSTRACT. Let C be a bounded closed convex subset of a Banach space X . It is shown that, in the category sense, almost all 1-set-contractions $f: C \rightarrow C$ are condensing. To know how the condensing mappings are scattered in the set $\Sigma_1(C)$ of 1-set-contractions it is proved that the set of noncondensing mappings is dense in $\Sigma_1(C)$.

1. Introduction. Throughout this paper X will denote a Banach space, C a closed convex bounded subset of X and $\Sigma_k(C)$ for some $k > 0$ the complete metric space of all k -set-contractions from C into C , endowed with the metric of uniform convergence. If f belongs to $\Sigma_k(C)$ ($k < 1$), Darbo [2] proved that f has a fixed point. Sadovskii [7] proved the same property for $f \in \Sigma_1(C)$ condensing (see definition below). Although apparently Sadovskii's Theorem is only a slight generalization of Darbo's Theorem, the real situation is very different. In fact, it is proved in this paper (Theorem 1) that almost all 1-set-contractions (in the sense of category) are condensing. However the set of all k -set-contractions ($k < 1$) is of Baire first category in $\Sigma_1(C)$ (Corollary 2). So, having in mind Sadovskii's Theorem, the generic existence of fixed points for 1-set-contractions proved in [1, 3, 5] is a straightforward consequence of Theorem 1. Furthermore the generic properness of $I - f$ (I the identity) for $f \in \Sigma_1(C)$ [5] is also obvious from Theorem 1. In order to better know how the set $\mathcal{C}$ of condensing mappings is scattered in $\Sigma_1(C)$, it is proved in Theorem 2 that $\Sigma_1(C) \setminus \mathcal{C}$ is dense in $\Sigma_1(C)$.

Recall that a mapping $f: C \rightarrow C$ is said to be a k -set-contraction if it is continuous and $\alpha(f(A)) \leq k\alpha(A)$ for every subset A of C , where $\alpha(\cdot)$ is the Kuratowski noncompactness measure [6, p. 412]. A mapping $f: C \rightarrow C$ is said to be condensing if it is continuous and $\alpha(f(A)) < \alpha(A)$ for every nonprecompact subset A of C . Finally, in a metric space (E, d) , $B(A, r)$ denotes for $A \subset E$ the set $\{x \in E: d(x, A) < r\}$. In a linear topological space $\text{co } A$ will mean the closed convex hull of A .

2. Generic results. A property is said to be generic in a Baire space E if it holds in a residual subset of E . In this section it will be proved that the 1-set-contractions are generically condensing.

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THEOREM 1. *The set $\mathfrak{C}$ of all condensing mappings in $\Sigma_1(C)$ is residual in $\Sigma_1(C)$.*

PROOF. Let N be the set of all k -set-contractions ($k < 1$) in $\Sigma_1(C)$; i.e. $N = \bigcup_{k < 1} \Sigma_k(C)$. For each $f \in N$ we denote by k_f the α -modulus of f . It is well known that N is a dense subset of $\Sigma_1(C)$. Define the set

$$\mathfrak{C}^* = \bigcap_{n=1}^{\infty} \bigcup_{f \in N} B(f, (1 - k_f)/2n).$$

Then $\mathfrak{C}^*$ is a dense G_δ -subset of $\Sigma_1(C)$. We claim that $\mathfrak{C}^*$ is contained in $\mathfrak{C}$. Let $g \in \mathfrak{C}^*$ and $A \subset C$ with $\alpha(A) > 0$. Choose a positive integer n such that $1/n < \alpha(A)$. Since g is in $\mathfrak{C}^*$, there exists $f \in N$ such that g is in $B(f, (1 - k_f)/2n)$ which implies, using the properties of $\alpha(\cdot)$, that

$$\alpha(g(A)) \leq \alpha(f(A)) + (1 - k_f)/n < k_f \alpha(A) + (1 - k_f) \alpha(A) = \alpha(A).$$

Hence g belongs to $\mathfrak{C}$.

COROLLARY 1 [1, 3, 5]. *There exists a residual subset $\mathfrak{C}$ of $\Sigma_1(C)$ such that for every $f \in \mathfrak{C}$ one has:*

- (i) *The set $\{x \in C: f(x) = x\}$ is compact and nonempty.*
- (ii) *If $\{f_n\} \rightarrow f$ in $\Sigma_1(C)$ and x_n is a fixed point of f_n , then there exists a subsequence of $\{x_n\}$ converging to a fixed point of f .*
- (iii) *For each $x \in C$, $\{f^n(x)\}$ is precompact.*

PROOF. (i) is obvious from Theorem 1 and Sadovskii's Theorem. To prove (ii) assume by contradiction that $\{x_n: n \geq 1\}$ is not a precompact set. Then

$$\alpha(f(\{x_n: n \geq 1\})) < \alpha(\{x_n: n \geq 1\})$$

and choose $\varepsilon > 0$, $\varepsilon < (\alpha(\{x_n: n \geq 1\}) - \alpha(f(\{x_n: n \geq 1\}))) / 2$. There exists an integer n_0 such that $d(f, f_n) < \varepsilon$ for every $n \geq n_0$. Then

$$\begin{aligned} \alpha(\{x_n: n \geq 1\}) &= \alpha(\{f_n(x_n): n \geq 1\}) = \alpha(\{f_n(x_n): n \geq n_0\}) \\ &\leq \alpha(\{f(x_n): n \geq n_0\}) + 2\varepsilon = \alpha(\{f(x_n): n \geq 1\}) + 2\varepsilon < \alpha(\{x_n: n \geq 1\}), \end{aligned}$$

which is a contradiction. A simpler argument proves (iii).

3. Density results.

THEOREM 2. *Assume that the set C is noncompact. Let f be a mapping in $\mathfrak{C}$ and $\varepsilon > 0$. There exists a mapping g in $\Sigma_1(C)$ such that $d(f, g) < \varepsilon$ and g is not condensing.*

PROOF. Let x_0 be a fixed point of f . We assume without loss of generality that $x_0 = 0$. Since C is a convex and noncompact set, $B(0, r) \cap C$ is noncompact for every $r > 0$. Choose $\delta > 0$, $\delta < \varepsilon/2$, such that $|f(x) - f(0)| < \varepsilon/2$ if $|x| \leq \delta$. Define a mapping

$$g(x) = \begin{cases} x & \text{if } x \in B(0, \delta/2) \cap C, \\ f(x) & \text{if } x \in C \setminus B(0, \delta), \\ t_x x + (1 - t_x) f(x) & \text{if } x \in B(0, \delta) \setminus B(0, \delta/2), \end{cases}$$

where $t_x = 2\delta^{-1}(\delta - |x|)$. It is easy to check that g is continuous and $d(f, g) < \varepsilon$. Furthermore g is not condensing because $B(0, \delta/2) \cap C$ is a noncompact set. We claim that g belongs to $\Sigma_1(C)$. Indeed, let A be a bounded subset of C and denote $A_1 = A \cap B(0, \delta/2)$, $A_2 = A \cap (B(0, \delta) \setminus B(0, \delta/2))$, $A_3 = A \cap (C \setminus B(0, \delta))$. Then $A = A_1 \cup A_2 \cup A_3$ and $\alpha(g(A)) = \max\{\alpha(g(A_i)): i = 1, 2, 3\}$. Since g is the identity in A_1 one has $\alpha(g(A_1)) = \alpha(A_1) \leq \alpha(A)$. Since $g = f$ in A_3 one has $\alpha(g(A_3)) = \alpha(f(A_3)) \leq \alpha(A_3) \leq \alpha(A)$. Finally if $x \in A_2$ one has $g(x) \in \text{co}(\{x, f(x)\}) \subset \text{co}(A_2 \cup f(A_2))$. Thus $g(A_2)$ is contained in $\text{co}(A_2 \cup f(A_2))$ which implies $\alpha(g(A_2)) \leq \max\{\alpha(A_2), \alpha(f(A_2))\} = \alpha(A_2) \leq \alpha(A)$.

COROLLARY 2. *Assume that the set C is noncompact. Then:*

- (i) *The set $\Sigma_1(C) \setminus \mathfrak{C}$ is dense in $\Sigma_1(C)$.*
- (ii) *The set $N = \bigcup_{k < 1} \Sigma_k(C)$ is of Baire first category in $\Sigma_1(C)$.*

PROOF. (i) is obvious from Theorem 2. To prove (ii) write $k_n = n/(n+1)$ and note that $N = \bigcup_{n=1}^{\infty} \Sigma_{k_n}(C)$ and for every k_n the closed set $\Sigma_{k_n}(C)$ is nowhere dense by Theorem 2.

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