CONTINUOUS FUNCTIONS ON POLAR SETS

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ABSTRACT. Let Ω be a second countable Brelot harmonic space with a positive potential. If K is a compact subset of Ω with more than one point, then K is a polar set iff every positive continuous function on K can be extended to a continuous potential on Ω . This is a generalization of the result proved by H. Wallin for the special case $\Omega = \mathbb{R}^n$ ($n \geq 3$) with Laplace harmonic space.

The objective of this paper is to generalize to axiomatic spaces of harmonic functions a well-known result of H. Wallin [6] which states that a compact set $K \subset \mathbb{R}^n$, $n \geq 3$, has Newtonian capacity 0 iff every positive continuous function on K is the restriction to K of a positive Newtonian potential on the whole space. We give the generalization of this result at the setting of Brelot space [3], which contains a positive potential, and the topology of the space has countable space for open sets. Accordingly, we prove

THEOREM 1. *Let K be a compact subset contained in Ω , a Brelot space as above, such that every positive continuous function on K can be uniformly approximated on K by positive superharmonic functions on Ω . Further, let K contain at least two points. Then K is polar.*

THEOREM 2. *Let K be a compact polar subset of Ω , f_0 a positive continuous function on K , and F_0 a relatively compact open neighborhood of K . Then there is a continuous potential p on Ω such that $p = f_0$ on K and p is harmonic on $\Omega \setminus \bar{F}_0$.*

In view of the fact that there are Brelot spaces of the nature mentioned here in which point sets need not be polar [2], the restriction in Theorem 1 is quite warranted. We can write $K = F \cup G$, with F and G closed subsets of Ω , neither of which is a subset of the other. Let z be in G and not in F . Then, by the Tietze extension theorem, for each positive integer n there is a continuous function f_n on K such that $f_n > 0$ on K , and $f_n(x) = 1$ for all x in F and $f_n(z) = 1/2^n$. Choose ϵ such that $0 < \epsilon < 1/2$. Then, by hypothesis, there is a positive superharmonic function q_n on Ω such that $|f_n(x) - q_n(x)| < \epsilon/2^{n-1}$, i.e.,

$$1 - \frac{\epsilon}{2^{n-1}} < q_n(x) < 1 + \frac{\epsilon}{2^{n-1}} \quad \text{for all } x \text{ in } F,$$

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and

$$\frac{1}{2^n} - \frac{\varepsilon}{2^{n-1}} < q_n(z) < \frac{1}{2^n} + \frac{\varepsilon}{2^{n-1}}.$$

Put $q(x) = \sum_{n=1}^{\infty} q_n(x)$ on Ω . Then, obviously, q is a superharmonic function and $q = \infty$ on F . Hence, F is a polar set. Similarly, G is also a polar set. Thus, K is a polar set.

The proof of Theorem 2 relies on first providing an approximation which is in line with the conclusion of the Theorem 1. We also include the following application, which is a generalization of another result of H. Wallin (see [6, Remark 4, p. 62]).

THEOREM 3. *Let Ω be a selfadjoint Brelot space [5] with second axiom of countability, δ a regular domain in Ω , and K a compact polar set $\subset \partial\delta$. Assume that the constant function 1 is superharmonic on Ω . Then, given any positive continuous function f on K , there is a continuous potential p on Ω such that p is harmonic on δ , $f = p$ on K , and $D_p(\delta) < \infty$, where D_p is the gradient measure, in the sense of Maeda, associated with p [5].*

PROOF. By Theorem 2 there is a continuous potential q_1 on Ω such that $f = q_1$ on K . Since f is continuous on K , there is a constant b such that $0 < f < b$ on K . Put $q = \min(q_1, b)$ on Ω . Then q is a bounded continuous potential on Ω and $q = f$ on K . Let

$$p(x) = \begin{cases} \int q(z) d\rho_x^\delta(z) & \text{if } x \in \delta, \\ q(x) & \text{if } x \in \Omega \setminus \delta. \end{cases}$$

Then p is a bounded continuous potential on Ω , and p is harmonic on δ . By a result of Maeda [5], $D_p(\delta) < \infty$. Further, since $K \subset \partial\delta$, $f(x) = q(x) = p(x)$ for every x in K . The proof is complete.

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1. Preliminaries. From now on, let K denote a compact polar set in a Brelot space Ω . The next theorem is a converse of Theorem 1.

THEOREM 4. *Given a positive continuous function on K and an $\varepsilon > 0$, there exists a continuous potential p on Ω such that $|f - p| < \varepsilon$ on K .*

PROOF. Let $\{U_n\}_{n=1}^{\infty}$ be a decreasing sequence of relatively compact open subsets of Ω such that $U_n \supset \overline{U_{n+1}} \supset U_{n+1}$ for $n = 1, 2, 3, \dots$, and $\bigcap_{n=1}^{\infty} U_n = K$. For each n , let f_n be a continuous extension of f to Ω such that $f_n \geq 0$ on Ω , and f_n has support contained in U_n . Taking the infimum at each stage, we may assume that $\{f_n\}_{n=1}^{\infty}$ is a decreasing sequence. Note that $f_n \rightarrow 0$ outside K and $f_n \rightarrow f$ on K . From the fact that $\{R_{f_n}^{U_n}\}_{n=1}^{\infty} = \{Rf_n\}_{n=1}^{\infty}$ is a decreasing sequence, it can be proved that $Rf_n \rightarrow R_f^K$ pointwise on Ω . However, it is routine to prove that $R_f^K = f$ on K . Hence, $Rf_n \rightarrow f$ on

K . Since, for each n , Rf_n is a continuous potential on Ω [2], by Dini's theorem, the convergence is uniform on K . Thus, there is an m such that $|Rf_n - f| < \varepsilon$ on K if $n \geq m$.

THEOREM 5. *Let f_0 be a positive continuous function on K , and let F_0 be a relatively compact open neighbourhood of K . Put $F = \bar{F}_0$ and let f be a nonnegative continuous extension of f_0 to Ω , such that $f > 0$ on F . Then, given $\varepsilon > 0$, there is a continuous potential p on Ω such that $p < f$ on F and $p \geq f_0 - \varepsilon$ on K . In fact, p can be chosen to be harmonic outside F .*

PROOF. We can assume $\varepsilon > 0$ is small enough so that $f_0 - \varepsilon > 0$ on K . Apply the proof of the previous theorem to the function $f_0 - \varepsilon$. Let U_n be a sequence of relatively compact open subsets of Ω such that $U_n \supset \bar{U}_{n+1} \supset U_{n+1}$, $n = 1, 2, 3, \dots$. Let g_n be a nonnegative, continuous extension of $f_0 - \varepsilon$ to Ω , with the support of $g_n \subset U_n$, and such that Rg_n converges to $(f_0 - \varepsilon)\chi_K$ on Ω as $n \rightarrow \infty$.

Now, let $\eta > 0$ such that $\eta < \varepsilon$ and $f_0 - \varepsilon + \eta < f_0$ on K . Since Rg_n converges to $f_0 - \varepsilon$ uniformly on K , there is an integer n_1 such that $Rg_n < f_0 - \varepsilon + \eta$ on K if $n \geq n_1$. Since Rg_n and $f - \varepsilon + \eta$ are two continuous functions on Ω , the set V , defined as $\{x \text{ in } \Omega: Rg_n(x) < f(x) - \varepsilon + \eta\}$, is an open set containing K . Further, Rg_n is a decreasing sequence of functions on Ω . Hence,

$$(1) \quad Rg_n(x) < f(x) - \varepsilon + \eta \quad \text{for every } x \text{ in } V \text{ and every } n \geq n_1.$$

Now $F_0 \cap V$ is an open set which clearly contains K . This implies that there is an integer n_2 such that

$$(2) \quad \bar{U}_n \subset V \cap F \quad \text{if } n \geq n_2.$$

Put $n_3 = \max(n_1, n_2)$. Choose $\alpha > 0$ such that $f \geq \alpha$ on F . Observe that $F \setminus U_{n_3}$ is a compact subset of Ω , and Rg_n decreases to zero on $F \setminus U_{n_3}$ as $n \rightarrow \infty$. Hence, by Dini's theorem, the convergence is uniform, i.e., there is an integer n_4 such that $Rg_n \leq \alpha/2$ on $F \setminus U_n$ if $n \geq n_4$.

Put $m = \max(n_3, n_4)$ and $p = Rg_m$. Then it follows that p is a continuous potential; on $F \setminus U_{n_3}$,

$$p = Rg_m \leq Rg_{n_4} < \alpha/2 < f;$$

and, lastly, using (1) and (2),

$$p = Rg_m \leq Rg_{n_3} \leq Rg_{n_1} \leq f - \varepsilon + \eta < f \quad \text{on } U_{n_3}.$$

Thus $p < f$ on F . Further, on K , $f_0 - \varepsilon = g_m \leq Rg_m = p$. The proof is complete.

2. Proof of Theorem 2. Now we prove the main theorem.

Put $F = \bar{F}_0$. Choose f , a nonnegative continuous extension of f_0 to Ω , such that $f > 0$ on F and f is zero outside a compact subset of Ω . Let $\{U_n\}_{n=0}^\infty$ be a sequence of relatively compact open subsets of Ω such that $U_{n+1} \subset \bar{U}_{n+1} \subset U_n$ for $n = 0, 1, 2, \dots$, and $K = \bigcap_{n=0}^\infty U_n$.

We can assume that $U_0 \subset F_0$. Fix a small $\varepsilon > 0$.

By the previous theorem there is a continuous potential p_0 on Ω such that: (i) $p_0(x) < f(x)$ for every x in F , (ii) $p_0(x) \geq f_0(x) - \varepsilon$ for every x in K , and (iii) p_0

is a harmonic function on the open set $\Omega \setminus \bar{U}_0$. Let $f_1 = \max(f - p_0, 0)$ on Ω . Then f_1 is a continuous function on Ω and $f_1 > 0$ on F . Applying the previous theorem to f_1 and then proceeding inductively, we obtain a sequence $\{p_n\}_{n=1}^\infty$ of continuous potentials on Ω such that: (i) p_i is a harmonic function on the open set $\Omega \setminus \bar{U}_i$ for every i , (ii) $\sum_{i=0}^n p_i < f$ on F for every n , and (iii) $\sum_{i=0}^n p_i \geq f_n - \varepsilon/2^n$ on K for every n .

Put $p = \sum_{i=0}^\infty p_i$. It is clear that p is a potential function on Ω . Further, on K ,

$$f_0 - \varepsilon/2^n \leq \sum_{i=0}^n p_i < f = f_0 \quad \text{for every } n.$$

Taking the limit as $n \rightarrow \infty$, we conclude that $p = f_0$ on K .

Now p_i is a harmonic function outside $\bar{U}_i \subset F_0 \subset F$ for every i . Hence, $\sum_{i=0}^n p_i$ is a harmonic function outside F . Therefore, on every connected component of $\Omega \setminus F$, $p = \lim_{n \rightarrow \infty} \sum_{i=0}^n p_i$ is either a harmonic function or identically $+\infty$. But p is a superharmonic function on Ω . This implies that p is finite on a dense subset of Ω , which eliminates the second possibility. Hence, p is a harmonic function on $\Omega \setminus F$. Thus, we are left with proving the continuity of p on Ω .

Let z be in $\Omega \setminus F$. Since p is a harmonic function on the open set $\Omega \setminus F$, it is continuous at z . Let z be in K . Then

$$\begin{aligned} p(z) &\leq \liminf_{x \rightarrow z} p(x) && \text{(by lower semicontinuity of } p) \\ &\leq \limsup_{x \rightarrow z} p(x) \\ &= \limsup_{x \rightarrow z, x \in F_0} p(x) && \text{(as } F_0 \text{ is an open neighbourhood of } z) \\ &\leq f(z) && \text{(as } f \text{ is continuous at } z) \\ &= f_0(z) \\ &= p(z) && \text{(as } p = f_0 \text{ on } K). \end{aligned}$$

Hence, p is continuous at z .

Now, let z in $F \setminus K$. Then there exists an integer N such that $z \notin U_n$ if $n \geq N$. By writing $p(x) = \sum_{i=0}^{N-1} p_i(x) + \sum_{i=N}^\infty p_i(x)$, we see that $\sum_{i=N}^\infty p_i(x)$ is a harmonic function in an open neighbourhood of z . In particular, $\sum_{i=N}^\infty p_i(x)$ is continuous at z . Being a finite sum of continuous functions at z , $\sum_{i=0}^{N-1} p_i$ is a continuous function at z . Hence, p is continuous at z . Thus, p is continuous on Ω , and the proof is complete.

COROLLARY 6. Assume that the constant functions are harmonic on Ω . If K is a compact polar subset of Ω , then every continuous function on K can be extended to a superharmonic function on Ω .

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