THE RADIAL BEHAVIOR OF BLASCHKE PRODUCTS

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ABSTRACT. The paper extends results of G. T. Cargo and of C. L. Belna, F. W. Carroll and G. Piranian on the existence of Blaschke products with prescribed radial limits at a prescribed finite or countable set of points on the unit circle. The extension permits the prescription of the radial cluster set at each point of a prescribed countable set on the unit circle.

1. Introduction. G. T. Cargo [2, p. 289] proved that if $\{\zeta_m\}_1^n$ and $\{w_m\}_1^n$ are a set of finitely many distinct points on the unit circle C and a set of points in the closure of the unit disk D , then some Blaschke product has the radial limit w_m at ζ_m for $m = 1, \dots, n$. He asked whether the theorem permits an extension to countably infinite sets $\{\zeta_m\}$ and $\{w_m\}$. Our paper shows that the answer is affirmative, and it goes beyond Cargo's question: it deals not merely with radial limits, but with radial cluster sets.

THEOREM. Let $M = \{\zeta_m\}$ and $\{K_m\}$ denote a countable set of distinct points on C and a sequence of nonempty, closed, connected sets in $D \cup C$. Then some Blaschke product has the radial cluster set K_m at ζ_m , for $m = 1, 2, \dots$.

2. The simplest case. We begin with the special case where M consists of the single point $\zeta = 1$ and the continuum $K_1 = K$ consists of a segment on the interval $[0, 1]$ of the real axis.

Even in the most general case, our proof will depend on elementary properties of the atomic singular function

$$A(z) = \exp[(z + 1)/(z - 1)]$$

and the composite function $B_0 = S \circ A$, where

$$S(w) = \frac{1/e - w}{1 - w/e} = \frac{1 - ew}{e - w}.$$

At the point 1, the function A has the radial limit 0 and therefore B_0 has the radial limit $1/e$. At all other points of C , the function A has a radial limit of modulus 1 and therefore B_0 has a radial limit of modulus 1. Because B_0 is a bounded function with a radial limit of modulus 1 almost everywhere on C , it is an inner function; because its radial limit is nowhere 0, it is a Blaschke product.

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The zeros of B_0 are the $1/e$ -points of A , in other words, the points z where the real and imaginary parts of $(z + 1)/(z - 1)$ are equal to -1 and to a multiple of 2π , respectively. The real part is equal to -1 everywhere on the circular arc $\{|z - 1/2| = 1/2, z \neq 1\}$ and nowhere else. We obtain information on the geometric distribution of the zeros of B_0 on this arc by solving the equation

$$(z_\nu + 1)/(z_\nu - 1) = -1 + 2\nu\pi i \quad (\nu = 0, \pm 1, \pm 2, \dots):$$

because the mapping $z \rightarrow (z + 1)/(z - 1)$ is an involution,

$$z_\nu = \frac{(-1 + 2\nu\pi i) + 1}{(-1 + 2\nu\pi i) - 1} = \frac{\nu\pi i}{\nu\pi i - 1}.$$

Obviously, $z_0 = 0$ and $z_\nu = 1/(1 + i/\nu\pi)$ ($\nu \neq 0$). It follows that if $\nu \neq 0$, then

$$z_\nu = 1 - i/\nu\pi - 1/(\nu\pi)^2 + i/(\nu\pi)^3 + O(\nu^{-4});$$

simple calculations now show that

$$|z_\nu|^2 = (1 - (\nu\pi)^{-2})^2 + (\nu\pi)^{-2} + O(\nu^{-4})$$

and therefore $|z_\nu| = 1 - 1/2(\nu\pi)^2 + O(\nu^{-4})$.

Suppose that corresponding to some number ρ ($0 < \rho < 1$) we have mutilated the Blaschke product B_0 by suppressing the finitely many factors whose zeros lie outside of the disk $\{|z - 1| < \rho\}$. (For the sake of brevity, we shall henceforth speak of suppression of certain zeros rather than suppression of the factors having these zeros.) By virtue of our estimate of $|z_\nu|$, the sum of the distances from the surviving zeros to the unit circle C is approximately

$$2 \sum_{\nu\pi > \rho} 1/2(\nu\pi)^2 = \pi^{-2} \sum_{\nu > \pi/\rho} \nu^{-2} \sim \rho/\pi^3.$$

On the other hand, because the set of suppressed zeros is finite and is symmetric relative to the real axis, the mutilated product has at $z = 1$ the same radial limit as B_0 , namely $1/e$. If corresponding to $\mu = 1, 2, \dots$ we obtain B_μ from B_0 by a finite and symmetric mutilation such that the sum of the distances from the surviving zeros to C is less than μ^{-2} , then the formal infinite product $B = \prod B_\mu$ is again a Blaschke product, and at $z = 1$ it has the radial limit 0. By means of further mutilations, we shall refine our control over the behavior of B on the segment $[0, 1)$.

Corresponding to each positive integer j and the integers $k = 0, 1, \dots, j - 1$, we consider the functions

$$A_{jk}(z) = \exp \frac{1}{j} \left(\frac{z + 1}{z - 1} - 2k\pi i \right) = A_{j0}(z) \exp \left(- \frac{2k\pi i}{j} \right).$$

Because $|A_{jk}(0)| = e^{-1/j}$, the functions A_{jk} have the constant modulus $e^{-1/j}$ on the circular arc $\{|z - 1/2| = 1/2, z \neq 1\}$. Clearly, A_{jk} assumes the real value $e^{-1/j}$ at the points on this arc where the imaginary part of $(z + 1)/(z - 1) - 2k\pi i$ is a multiple of $2j\pi$. The function

$$B_{jk} = (1 - e^{1/j} A_{jk}) / (e^{1/j} - A_{jk})$$

has the radial limit 0 nowhere on C ; therefore it is a Blaschke product. Obviously, its zeros (the $e^{-1/j}$ -points of A_{jk}) are the zeros z_ν of B_0 whose indices satisfy the condition $\nu \equiv k \pmod{j}$.

Now let $\{\gamma_k\}$ denote a sequence of numbers in the interval $(0, 1)$ such that the segment K is the set of limit points of $\{\gamma_k\}$. For technical reasons, we also require that $\gamma_1 > e^{-1}$ and that $\log \gamma_k$ is rational and $0 < |\log \gamma_{k+1}/\gamma_k| < 1$ for $k = 1, 2, \dots$.

We form a periodic sequence $\{\nu_{1i}\}_{i=1}^\infty$ of density $-\log \gamma_1$ and denote by B_1 the Blaschke product obtained from B_0 by suppression of all the zeros z_ν except those for which $|\nu| \in \{\nu_{1i}\}$. From our earlier discussion it follows that at $z = 1$ the function B_1 has the radial limit γ_1 . We can therefore choose a number r_1 in $(0, 1)$ such that

$$|B_1(x) - \gamma_1| < 2^{-1} \quad \text{whenever } r_1 \leq x < 1.$$

Our next step depends on whether $\gamma_2 < \gamma_1$ or $\gamma_2 > \gamma_1$.

If $\gamma_2 < \gamma_1$, we form a periodic sequence $\{\nu_{2i}\}$ of density $-\log \gamma_2/\gamma_1$ and denote by B_2 the Blaschke product obtained from B_0 by suppression of all the zeros z_ν except those for which $|\nu| \in \{\nu_{2i}\}$. Then the radial limit of $B_1 B_2$ at $z = 1$ is γ_2 ; moreover, if the initial element of $\{\nu_{2i}\}$ is sufficiently large, then $|B_2(x) - 1| < 2^{-2}$ whenever $0 < x < r_1$.

If $\gamma_2 > \gamma_1$, we denote by B_2 the reciprocal of a Blaschke product whose zeros occur in B_1 and are selected so that $B_1 B_2$ has at 1 the radial limit γ_2 . Again, if the sequence of selected zeros begins late enough, then $|B_2(x) - 1| < 2^{-2}$ when $0 < x < r_1$.

Suppose we have constructed a product $B_1 \cdots B_{k-1}$ whose behavior on $(0, 1)$ reflects the behavior of the finite sequence $\{\gamma_1, \dots, \gamma_{k-1}\}$.

If $\gamma_k < \gamma_{k-1}$, we construct the Blaschke product B_k so that $B_1 \cdots B_k$ has the radial limit γ_k at 1 and so that $|B_k(x) - 1| < 2^{-k}$ on some interval $(0, r_{k-1}]$ on which the behavior of $B_1 \cdots B_{k-1}$ also reflects the behavior of the finite sequence $\{\gamma_1, \dots, \gamma_{k-1}\}$.

If $\gamma_k > \gamma_{k-1}$, we denote by B_k the reciprocal of a Blaschke product whose zeros are zeros of $B_1 \cdots B_{k-1}$ and whose radial limit at $z = 1$ is γ_{k-1}/γ_k . To satisfy ourselves that the supply of zeros of $B_1 \cdots B_{k-1}$ is sufficient for our purpose, we observe that $0 < \gamma_k < 1$ and that if we suppressed all except finitely many of the zeros of the function $B_1 \cdots B_{k-1}$, the modulus of its radial limit at $z = 1$ would rise to the value 1.

If for $k = 1, 2, \dots$ the sequence of zeros (poles) of B_k begins near enough to C , then the formal product $B = \prod B_k$ is a Blaschke product and its radial cluster set at 1 is the segment K .

3. The unrestricted continuum. In this section, we suppose that $M = \{1\}$ and that $K_1 = K$ is a nonempty continuum in $D \cup C$. Again we choose a sequence $\{\gamma_n\}$ whose set of limit points is the continuum K . In the statement of our technical conditions on the sequence $\{\gamma_k\}$, we replace γ_k with $|\gamma_k|$ and impose the additional restriction that $\gamma_k/\gamma_{k+1} \rightarrow 1$ as $k \rightarrow \infty$. Repeating the construction described in §2, we obtain a Blaschke product B whose behavior on the segment $[0, 1)$ reflects with

progressively higher accuracy the behavior of the sequence $\{|\gamma_k|\}$. We shall now show that we can also control the argument of B on the segment $(0, 1)$.

To determine the contribution to $\arg B(x)$ from a single Blaschke factor with its zero at $re^{i\theta}$ ($0 < |\theta| < \pi$), we write the factor in the form

$$b(z) = e^{-i\theta}(re^{i\theta} - z)/(1 - re^{-i\theta}z).$$

If the zero lies on the circle $\{|z - 1/2| = 1/2\}$, then $r = \cos \theta$, and therefore $b(0) = r = \cos \theta$ and

$$b(1) = \frac{r - e^{-i\theta}}{1 - re^{-i\theta}} = \frac{i \sin \theta}{\sin^2 \theta + i \cos \theta \sin \theta} = e^{i\theta}.$$

Because the function b maps the segment $[0, 1]$ on the real axis onto a circular arc orthogonal to C , and because this arc has the endpoints $\cos \theta$ and $e^{i\theta}$, the argument of $b(x)$ changes monotonically from 0 to θ as x moves from 0 to 1 along the real axis.

Suppose we have constructed a finite product $B = B_1 \cdots B_n$ (where each factor is either a subproduct of B_0 or the reciprocal of such a subproduct) with the following three properties. (i) The intersection of some disk $|z - 1| < \delta_n$ with the set of zeros of B is symmetric with respect to the real axis, in the strict sense that zeros at z and at $\bar{z}$ have the same multiplicity. (ii) The zeros of B_0 in the upper half of the disk $|z - 1| < \delta_n$ that occur in B have indices forming a periodic sequence. (iii) The function B maps the segment $[0, 1]$ onto an arc that follows closely the polygonal arc with the vertices

$$\gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_{n-1}|\gamma_n/\gamma_{n-1}|.$$

From the discussion in the preceding paragraph it follows that the suppression of an additional zero z_k changes the argument of the radial limit at 1 of B by $-\arg z_k$. Because the arguments of the zeros (in the upper half-plane) of B_0 essentially form a harmonic sequence, we can choose a finite number of the surviving zeros of B in the upper or lower half-plane so that after their suppression the function B maps the segment $[1, 0]$ onto an arc following closely the polygonal arc with the vertices $\gamma_1, \gamma_2, \dots, \gamma_n$.

Next we adjoin to B either a new Blaschke product B_{n+1} (if $|\gamma_{n+1}| < |\gamma_n|$) or the reciprocal of a Blaschke product (if $|\gamma_{n+1}| > |\gamma_n|$). The continuation of the construction produces a Blaschke product whose radial cluster set at 1 is the continuum K .

4. The general case. If M consists of a single point ζ on C rather than the single point 1, we construct a Blaschke product B having the radial cluster set K at 1 and define a Blaschke product P by the formula $P(z) = B(z/\zeta)$.

If $\{\zeta_n\}$ is a finite or countable set on C and $\{K_m\}$ is a corresponding sequence of nonempty continua in $D \cup C$, we construct a sequence $\{P_m\}$ of Blaschke products such that for each index m the function P_m is holomorphic on C (except at the point ζ_m) and has at ζ_m the radial cluster set K_m .

If from each of the products P_m we form a product Q_m by suppressing a sufficiently large number of early zeros, the function $Q = \prod Q_m$ is a Blaschke product with the property that for each index m the subproduct Q/Q_m has at ζ_m a

radial limit of modulus 1. The radial cluster set of Q at ζ_m then has the form $e^{i\phi_m}K_m$, where ϕ_m , the angle of rotation, depends not only on the radial limit at ζ of Q/Q_m , but also on the set of early zeros stripped from P_m in the passage to Q_m . By Theorem 2 in [1], there exists a Blaschke product Q_0 whose radial limit at ζ_m is $e^{-i\phi_m}$ ($m = 1, 2, \dots$). Clearly, the Blaschke product Q_0Q has the properties promised in our theorem.

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