ON THE SPECTRAL MULTIPLICITY OF A CLASS OF FINITE RANK TRANSFORMATIONS

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ABSTRACT. The rank M transformations, which Chacon called the simple approximations with multiplicity M , were shown by Chacon to have maximal spectral multiplicity at most M , although no example was given where this bound is attained for $M > 1$. In this paper, for each natural number $M > 1$, we show how to construct a simple approximation with multiplicity M which is ergodic and has maximal spectral multiplicity equal to $M - 1$.

Introduction. The purpose of this paper is to give a method of constructing ergodic automorphisms $\tau: [0, 1) \rightarrow [0, 1)$ with finite rank and maximal spectral multiplicity greater than one. In particular for each natural number $M > 1$ we show how to construct ergodic automorphisms which admit simple approximations with multiplicity M (in the sense of Chacon [2]) and have maximal spectral multiplicity equal to $M - 1$.

Oseledec [5], using interval exchange transformations, was the first to construct an ergodic automorphism with finite spectral multiplicity greater than one. His example involved an exchange of 30 intervals and had maximal spectral multiplicity lying between 2 and 30. Robinson [7] and Katok (unpublished) generalised his result again using interval exchange transformations. Katok constructed an example with multiplicity equal to 2 and Robinson constructed ergodic automorphisms with arbitrary finite maximal spectral multiplicity. Our examples are constructed using the stacking method with M -columns; they are ergodic having discrete spectrum on an invariant subspace and continuous spectrum with multiplicity $M - 1$ on its orthogonal complement. The main significance of the construction is its simplicity. The construction is based on an example of del Junco [3] and Baxter [1].

1. Preliminaries. Let $\tau: (X, \mathfrak{B}, \mu) \rightarrow (X, \mathfrak{B}, \mu)$ be an invertible measure preserving transformation of a Lebesgue space $(X, \mathfrak{B}, \mu)$, and let $U_\tau: L^2(X) \rightarrow L^2(X)$ be the induced unitary operator defined by $(U_\tau f)(x) = f(\tau^{-1}x)$. Then the *maximal spectral multiplicity* of U_τ is defined as follows.

DEFINITION 1. Let $U: H \rightarrow H$ be a unitary operator on a Hilbert space H . The *maximal spectral multiplicity* of U is $m(U) = \inf\{m \in \mathbf{Z}^+ \cup \{\infty\} : \exists f_1, \dots, f_m \in H \text{ such that linear combinations of } U^i f_j, i \in \mathbf{Z}, j = 1, \dots, m, \text{ are dense in } H\}$.

When the multiplicity is one we say the spectrum is *simple*. If, for U_τ , the eigenvalue 1 is simple and is the only eigenvalue, we say τ has *continuous spectrum*. (See Robinson [7] and Parry [6] for more details and a discussion of the history of the spectral theory of measure preserving transformations.)

DEFINITION 2. A *partition* of X is a finite collection $\zeta = \{A_i : i \in I, A_i \in \mathfrak{B}\}$ of disjoint sets. A τ -*stack* S is a partition $\{S_0, \dots, S_{n-1}\}$ of X such that $\tau S_i = S_{i+1}$

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for $0 \leq i < n-1$. S_0 is called the *base* of S (written $B(S)$), S_{n-1} the *top* and n the *height*.

2. Construction of τ and the main theorem. To illustrate the general method and simplify the computations we construct an ergodic automorphism τ which admits simple approximations with multiplicity 3 and has maximal spectral multiplicity equal to 2. The general construction for M a natural number is then a straightforward generalization of this.

τ will be defined on $(X, \mathfrak{B}, \mu)$, where $X = [0, 1)$, $\mathfrak{B}$ is the σ -algebra of Borel sets and μ is the Lebesgue measure. We proceed by the stacking method using three stacks. At the n th stage we will have three τ -stacks S_n^0 , S_n^1 and S_n^2 whose levels are intervals of length $1/3^{n+1}$ and τ will map each level in S_n^i except the top one linearly onto the level directly above. τ will be undefined on the top of S_n^i .

We define S_n^i inductively, first putting $S_0^0 = [0, \frac{1}{3})$, $S_0^1 = [\frac{1}{3}, \frac{2}{3})$ and $S_0^2 = [\frac{2}{3}, 1)$. Suppose S_n^0 , S_n^1 and S_n^2 have been constructed and have height 3^n . Cut $B(S_n^i)$ into three intervals I_n^i , J_n^i and K_n^i of equal length and let τ map $\tau^{3^n-1}I_n^i$ linearly onto K_n^{i+1} and $\tau^{3^n-1}K_n^i$ linearly onto J_n^{i+2} ($i = 0, 1, 2$; where $i+1, i+2$ are interpreted mod 3).

This defines τ -stacks S_{n+1}^0 , S_{n+1}^1 , S_{n+1}^2 of height 3^{n+1} and bases I_n^0 , I_n^1 , I_n^2 , respectively. Throughout we shall denote the base of S_n^i by B_n^i , i.e. $B(S_n^i) = B_n^i$, $i = 0, 1, 2$.

We now state the main theorem. The major part of the proof follows from a number of propositions given in the next section.

THEOREM 1. *τ is an ergodic automorphism with maximal spectral multiplicity equal to two.*

PROOF. Regarding τ as a single stack with base $\bigcup_{i=0}^2 B_n^i$ and height 3^n , the ergodicity follows from the usual arguments for single stacks (see Friedman [4]). Note also that τ has as eigenvalues all the 3^n th roots of unity. The rest of the proof follows from Propositions 1, 2 and 3 of §3.

3. Invariant subspaces for U_τ and the proof of the theorem. We introduce invariant subspaces H_k , $k = 0, 1, 2$, in $L^2(X)$ by first defining $\sigma: X \rightarrow X$ by $\sigma(x) = x + \frac{1}{3} \pmod{1}$ and writing $H_k = \{f \in L^2(X): U_\sigma f = \omega^k f\}$, $k = 0, 1, 2$; where $\omega = e^{2\pi i/3}$ and $U_\sigma: L^2(X) \rightarrow L^2(X)$ is the operator $(U_\sigma f)(x) = f(\sigma^{-1}x)$.

LEMMA 1. (i) σ and τ commute.

(ii) H_k , $k = 0, 1, 2$, is invariant under U_τ and $L^2(X) = H_0 \oplus H_1 \oplus H_2$.

PROOF. (i) By induction we see that $\sigma(\tau^k B_n^0) = \tau^k(B_n^1)$, $\sigma(\tau^k B_n^1) = B_n^2$ and $\sigma(\tau^k B_n^2) = B_n^0$ for $k = 0, 1, \dots, 3^n - 1$; $n = 1, 2, \dots$. It follows that τ and σ commute.

(ii) From (i) H_k is invariant under U_τ , $k = 0, 1, 2$. Furthermore $L^2(X)$ has an orthonormal basis $\{\phi_n: n \in \mathbf{Z}\}$, where $\phi_n(z) = z^n$, $z \in K$ (identifying X with the unit circle K in the complex plane). Since $\phi_{3m+k} \in H_k$ for $m \in \mathbf{Z}$, $k = 0, 1, 2$, it follows that $L^2(X) = H_0 \oplus H_1 \oplus H_2$.

PROPOSITION 1. $U_\tau|_{H_0}$ has discrete spectrum with eigenvalues precisely all the 3^n th roots of unity.

PROOF. Denote $\mathfrak{B}_\zeta = \{E: \sigma E = E\}$ as the sub- σ -algebra of $\mathfrak{B}$ of sets invariant under σ , and let ζ be the corresponding partition of X . Note that $H_0 = L^2(X/\zeta)$ and the conditional expectation $E(\cdot|\zeta): L^2(X) \rightarrow L^2(X/\zeta)$ is the projection onto H_0 . Using these considerations it can be seen that $U_\tau|_{H_0}$ arises from the von Neumann-Kakutani type transformation obtained by cutting $[0, 1]$ into three intervals of equal length, stacking into a single column and repeating the process to obtain a column of height 3^n at the n th stage. (In face, τ acting on the σ -algebra $\mathfrak{B}_\zeta$ is precisely this transformation.) In particular, the eigenvalues are precisely all the 3^n th roots of unity; each eigenvalue is simple so $U_\tau|_{H_0}$ has simple spectrum.

PROPOSITION 2. $U_\tau|_{H_1}$ and $U_\tau|_{H_2}$ each have simple continuous spectrum.

PROOF. (i) For the continuity of spectrum of $U_\tau|_{H_1}$, let $f \in H_1$ and $\lambda \in K$ with $U_\tau f = \lambda f$. Then $U_\sigma f = \omega f$ so $U_\sigma f^3 = f^3$, or $f^3 \in H_0$. Now $U_\tau f^3 = \lambda^3 f^3$ and since $U_\tau|_{H_0}$ only has eigenvalues which are 3^n th roots of unity, λ^3 is a 3^n th root of unity and so also is λ . It follows that U_τ has two orthogonal eigenfunctions corresponding to the same eigenvalue, contradicting the ergodicity of τ . A similar argument applies to $U_\tau|_{H_2}$.

(ii) To see that $U_\tau|_{H_2}$ has simple spectrum write

$$f_n = \chi_{B_n^0} + \omega \chi_{\sigma B_n^0} + \omega^2 \chi_{\sigma^2 B_n^0} = \chi_{B_n^0} + \omega \chi_{B_n^1} + \omega^2 \chi_{B_n^2}.$$

Then

$$U_\sigma f_n = \chi_{\sigma B_n^0} + \omega \chi_{\sigma^2 B_n^0} + \omega^2 \chi_{\sigma^3 B_n^0} = \omega^2 f_n$$

so that $f_n \in H_2$. Clearly $\|f_n\| \rightarrow 0$ as $n \rightarrow \infty$. Let $Z(f_n)$ denote the cyclic subspace generated by f_n ; then we claim $Z(f_n) \nearrow H_2$ as $n \rightarrow \infty$. It follows from Baxter [1, Lemma 14] that $U_\tau|_{H_2}$ has simple spectrum. A similar argument using $g_n = \chi_{B_n^0} + \omega^2 \chi_{B_n^1} + \omega \chi_{B_n^2}$ implies $U_\tau|_{H_1}$ has simple spectrum.

To show that $Z(f_n) \nearrow H_2$ let $f \in L^2(X)$ and notice that the projection $E: L^2(X) \rightarrow H_2$ is defined by $E(f) = \frac{1}{3}(f + \omega U_\sigma f + \omega^2 U_{\sigma^2} f)$. It is sufficient to show that $E(f)$ can be approximated arbitrarily closely by linear combinations of $U_\tau^m(f_n)$, $m \in \mathbf{Z}$. Since the partition $\{\tau^j B_n^i: i = 0, 1, 2; j = 0, 1, \dots, 3^n - 1; n = 0, 1, \dots\}$ generates the Borel sets, there exists

$$h_n = \sum_{i=0}^2 \sum_{j=0}^{3^n-1} a_{ij} \chi_{\tau^j B_n^i}, \quad a_{ij} \in \mathbf{C}, n \in \mathbf{N},$$

satisfying $\|f - h_n\| < \epsilon$ or $\|E(f) - E(h_n)\| < \epsilon$. Now $E(h_n) \in Z(f_n)$ since

$$\begin{aligned} E(h_n) &= \sum_{i=0}^2 \sum_{j=0}^{3^n-1} a_{ij} \frac{1}{3} (\chi_{\tau^j B_n^i} + \omega \chi_{\tau^j \sigma B_n^i} + \omega^2 \chi_{\tau^j \sigma^2 B_n^i}) \\ &= \frac{1}{3} \sum_{i=0}^2 \sum_{j=0}^{3^n-1} a_{ij} U_\tau^j (\chi_{B_n^i} + \omega \chi_{\sigma B_n^i} + \omega^2 \chi_{\sigma^2 B_n^i}) \\ &= \frac{1}{3} \sum_{i=0}^2 \sum_{j=0}^{3^n-1} \omega^{-i} a_{ij} U_\tau^j (f_n) \in Z(f_n), \end{aligned}$$

and the result follows.

PROPOSITION 3. $U_\tau|H_1 \oplus H_2$ has maximal spectral multiplicity equal to two.

PROOF. $U_\tau|H_i, i = 1, 2$, each have simple continuous spectrum, so the multiplicity of $U_\tau|H_1 \oplus H_2$ cannot exceed two. Suppose $U_\tau|H_1 \oplus H_2$ has multiplicity equal to one. Then there exists $w \in H_1 \oplus H_2$ with $Z(w) = H_1 \oplus H_2$. If f_n and g_n are as defined in Proposition 2, $f_n, g_n \in Z(w)$ and $Z(f_n) \perp Z(g_n)$. Denote the maximal spectral types of $U_\tau|Z(g_n)$ and $U_\tau|Z(f_n)$ by ρ_1^n and ρ_2^n , respectively. Then ρ_1^n and ρ_2^n are mutually singular (see Parry [6, p. 93]). The theorem will follow if we show $\rho_1^n = \rho_2^n, n \in \mathbf{N}$, and for this it suffices to show $(U_\tau^m f_n, f_n) = (U_\tau^m g_n, g_n)$ for all $m \in \mathbf{Z}, n \in \mathbf{N}$.

Now

$$\begin{aligned} (U_\tau^m f_n, f_n) &= \int U_\tau^m f_n \cdot \bar{f}_n d\mu \\ &= \int (\chi_{\tau^m B_n^0} + \omega \chi_{\tau^m \sigma B_n^0} + \omega^2 \chi_{\tau^m \sigma^2 B_n^0})(\chi_{B_n^0} + \omega^2 \chi_{\sigma B_n^0} + \omega \chi_{\sigma^2 B_n^0}) d\mu \\ &= 3\mu(\tau^m B_n^0 \cap B_n^0) + 3\omega^2 \mu(\tau^m B_n^0 \cap \sigma B_n^0) + 3\omega \mu(\tau^m B_n^0 \cap \sigma^2 B_n^0) \end{aligned}$$

since σ is measure preserving. A similar calculation shows that

$$(U_\tau^m g_n, g_n) = 3\mu(\tau^m B_n^0 \cap B_n^0) + 3\omega \mu(\tau^m B_n^0 \cap \sigma B_n^0) + 3\omega^2 \mu(\tau^m B_n^0 \cap \sigma^2 B_n^0)$$

for all $m \in \mathbf{Z}, n \in \mathbf{N}$. Thus the result follows on noting that $\mu(\tau^m B_n^0 \cap \sigma B_n^0) = \mu(\tau^m B_n^0 \cap \sigma^2 B_n^0)$ for all $m \in \mathbf{Z}, n \in \mathbf{N}$, as can be seen from the symmetry of the construction of τ .

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