A RIGIDITY RESULT FOR HOLOMORPHIC IMMERSIONS OF SURFACES IN CP^n

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ABSTRACT. A pinching condition for the Gaussian curvature implies rigidity.

In this paper we prove a stronger version of a rigidity result obtained by Lawson [3] (the quantization lemma). Toward this aim we briefly recall some results and formulas for holomorphic immersions of a Riemann surface into projective space; we refer the reader to [1-3] for details.

Let R be a Riemann surface and $f: R \rightarrow CP^n$ a holomorphic immersion, where CP^n is the complex n -dimensional projective space with the normalized Fubini-Study metric of constant holomorphic sectional curvature 1.

Let z be a local complex coordinate on R . We can express the metric induced by f on R as

$$(1) \quad ds^2 = 2F dz d\bar{z},$$

where F is a positive smooth function. Then the Laplace operator and the Gauss curvature K of this metric are respectively given by

$$(2) \quad \Delta = \frac{2}{F} \frac{d}{dz} \frac{d}{d\bar{z}},$$

$$(3) \quad K = -\frac{1}{F} \frac{d}{dz} \frac{d}{d\bar{z}} \log F.$$

If the image $f(R)$ in CP^n lies in no proper, totally geodesic submanifold of CP^n , then Calabi [2] showed that it is possible to define inductively a sequence of functions $\{F_k\}$, $k = 0, \dots, n+1$, by setting

$$(4) \quad \begin{cases} F_0 = 1, & F_1 = F, \\ F_{k+1} = \frac{F_k^2}{F_{k-1}} \left(\frac{d}{dz} \frac{d}{d\bar{z}} \log F_k + \frac{k+1}{2} F \right) \end{cases}$$

for $k = 1, \dots, n$ and they satisfy the following:

(5.i) For $0 \leq k \leq n$, F_k is nonnegative and vanishes only at isolated points.

(5.ii) The succeeding function F_{k+1} is defined by (4) away from those points but extends to a real analytic function on all of R .

(5.iii) $F_{n+1} \equiv 0$.

Using the sequence $\{F_k\}$, the curvature functions are inductively defined by

$$(5) \quad K_k = \frac{F_{k+1} F_{k-1}}{F_k^2} \quad \text{for } 1 \leq k \leq n.$$

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We remark that the curvature functions are independent of local coordinates and that $K_n \equiv 0$. Moreover $K_k \geq 0$ and, for $1 \leq k < n$, the set of zeros Z_{K_k} contains only isolated points.

From the recursive definition (5) of the curvature functions it is not hard to see that the following nontrivial relations hold:

$$(6) \quad K_1 = 1 - K,$$

$$(7) \quad \frac{1}{2} \Delta \log K_k = K_{k+1} + K_{k-1} - 2K_k + K \quad \text{for } k = 1, \dots, n-1,$$

where $K_0 = \frac{1}{2}$.

By induction from (6) and (7) we obtain

$$(8) \quad \Delta \log K_1^{n-1} K_2^{n-2}, \dots, K_{n-1} = n(n+1)(K - 1/n).$$

Of course formulas (7) and (8) are valid on their appropriate domains of definition.

Now suppose R to be compact. Then the integrated k th curvature ν_k is defined to be

$$(9) \quad \nu_k = \frac{1}{2\pi} \int_R K_k dA, \quad k = 0, \dots, n,$$

where dA is the area element of the Riemann surface R . It is well known that ν_k is an integer, precisely the order of the k th osculating curve of the isometric immersion of R into $\mathbf{CP}^n$. In particular, $4\pi\nu_0 = A$, the area of $f: R \rightarrow \mathbf{CP}^n$. Define σ_k to be

$$(10) \quad \sigma_k = -\frac{1}{4\pi} \int_R \Delta \log K_k dA, \quad k = 0, \dots, n.$$

The σ_k 's and the ν_k 's are related by the classical Plücker formulas

$$(11) \quad \sigma_k = 2\nu_k - \nu_{k+1} - \nu_{k-1} + 2(g-1), \quad k = 0, \dots, n-1.$$

where g is the genus of R and $\nu_{-1} = 0$. with C_n we will indicate Calabi's curve (see [1]) in $\mathbf{CP}^n$ of constant Gaussian curvature $1/n$. C_n does not lie in any linear subspace of $\mathbf{CP}^n$ and is given by the following imbedding of $\mathbf{CP}^1$ in $\mathbf{CP}^n$:

$$(12) \quad (z_0, z_1) \rightarrow \left(z_0^n, \sqrt{n} z_0^{n-1} z_1, \dots, \sqrt{\binom{n}{k}} z_0^{n-k} z_1^k, \dots, z_1^n \right).$$

The curvature functions for this curve are

$$(13) \quad K_k = \frac{(k+1)}{2n} (n-k).$$

Calabi [1] showed that, modulo holomorphic congruences, C_n is the only curve in $\mathbf{CP}^n$ of constant Gaussian curvature which does not lie in any linear subspace.

We are now ready to state and prove the following result.

THEOREM. *Let R be a compact, connected Riemann surface (without boundary) and let $f: R \rightarrow \mathbf{CP}^n$ be a holomorphic immersion such that $f(R)$ does not lie in any linear subspace of $\mathbf{CP}^n$. Let K be the Gaussian curvature of the induced metric. If*

$$(14) \quad K \geq 1/n,$$

then R is topologically a sphere, $K \equiv 1/n$ and $f(R) = C_n$, where C_n is defined above.

PROOF. First of all we observe that $Z_{K_1} \cup \dots \cup Z_{K_{n-1}} \neq R$. From (8) and (14) we obtain

$$(15) \quad \Delta \log K_1^{n-1} K_k^{k-2}, \dots, K_{n-1} \geq 0 \quad \text{on } R \setminus (Z_{K_1} \cup \dots \cup Z_{K_{n-1}}).$$

Now let p be a point of R in which $K_1^{n-1} \dots K_{n-1}$ attains its positive absolute maximum. Then (15) is valid in a neighborhood U of p and at p $\log K_1^{n-1} \dots K_{n-1}$ attains its absolute maximum. The maximum principle for subharmonic functions (see [5]) then applies to show that $\log K_1^{n-1} \dots K_{n-1}$, and hence $K_1^{n-1} \dots K_{n-1}$, is constant on U . A standard argument using connectedness of R then shows that

$$(16) \quad K_1^{n-1} \dots K_{n-1} = \text{constant on } R.$$

From (8) and (16) we deduce $K \equiv 1/n$. Then the Gauss-Bonnet theorem applies to show $g = 0$, completing the proof.

Suppose R to be compact. Then integrating (8) we obtain the following estimate for the area of the immersion:

$$(17) \quad A = 4\pi n(1 - g) + \frac{4\pi}{n+1} \sum_{k=1}^{n-1} (n-k)\sigma_k.$$

In particular, for a holomorphic immersion of a sphere such that $f(R)$ does not lie in any subspace of $\mathbf{CP}^n$ we have

$$(18) \quad A \geq 4\pi n.$$

Remark that if $Z_{K_k} = \emptyset$, then $\sigma_k = 0$ because of (10); actually the converse is true because of the following

LEMMA. *Let R be a compact Riemann surface, $h: R \rightarrow \mathbf{R}$ be a smooth positive function whose zeros set Z contains only isolated points and*

$$(19) \quad \Delta \log h = f \quad \text{on } R \setminus Z$$

for some continuous $f: R \rightarrow \mathbf{R}$. Then

$$(20) \quad \int_R \Delta \log h \leq 0$$

and equality holds if and only if $Z = \emptyset$.

We omit the computational proof of the lemma.

We have the following

PROPOSITION. *In the same hypothesis of the theorem, the following are equivalent:*

- (i) $\nu_0 \leq n(1 - g)$,
- (ii) $Z_{K_1} = \dots = Z_{K_{n-1}} = \emptyset$,
- (ii) $g = 0$, $\nu_k = (k+1)(n-k)$, $k = 0, \dots, n-1$.

PROOF. (i) $\rightarrow$ (ii). Indeed, since $A = 4\pi\nu_0$ from (i) and (17), we get $\sigma_k = 0$, $k = 1, \dots, n-1$; hence, by the above remark, (ii).

(ii)→(iii). From (ii) and the above remark, we have $\sigma_k = 0$, $k = 1, \dots, n-1$; hence from (11) we obtain the system of equations

$$(21) \quad 0 = 2\nu_k - \nu_{k+1} - \nu_{k-1} + 2(g-1), \quad k = 0, \dots, n-1.$$

It is easy to see that the solutions of (21) are given by

$$(22) \quad \nu_k = (k+1)(\nu_0 + k(g-1)), \quad k = 0, \dots, n,$$

and since $\nu_n = 0$ we obtain $\nu_0 = n(1-g)$. On the other hand $A = 4\pi\nu_0$ and we deduce, using the Gauss-Bonnet theorem, that $g = 0$ and hence (iii).

(iii)→(i). It is obvious.

REMARKS. (1) In the case $n = 2$ from (6) we immediately obtain from $K \leq \frac{1}{2}$ that (ii) holds, hence (i) and using the Gauss-Bonnet theorem we get $K \equiv \frac{1}{2}$. This case was considered by Nomizu and Smyth [4]. The previous proposition should be compared with Theorem 3 in Lawson [3].

(2) If we suppose

$$(23) \quad K_2 \geq 3 \left(\frac{1}{2} - K \right),$$

then from (7) we have $\Delta \log K_1 \geq 0$ and we deduce as in the theorem that K_1 , and hence K , is constant. Again we observe that in the case $n = 2$ (23) becomes $K \geq \frac{1}{2}$.

REFERENCES

1. E. Calabi, *Isometric imbeddings of complex manifolds*, Ann. of Math. (2) **58** (1953), 1-23.
2. —, *Metric Riemann surfaces*, Contributions to the Theory of Riemann Surfaces, Princeton Univ. Press, Princeton, N.J., 1953.
3. H. B. Lawson, *The Riemannian geometry of holomorphic curves*, Proc. Conf. on Holomorphic Mappings and Minimal Surfaces, Bol. Soc. Brasil. Mat. **2** (1971), 45-62.
4. K. Nomizu and B. Smyth, *Differential geometry of complex hypersurfaces*. II, J. Math. Soc. Japan **20** (1968), 498-521.
5. M. H. Protter and H. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

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63130