TOTALLY REAL MINIMAL SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE

FRANCISCO URBANO

ABSTRACT. An n -dimensional positively curved compact totally real minimal submanifold of an n -dimensional complex projective space is totally geodesic.

1. Statement of result. Let CP^n be an n -dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature c , and let M be an n -dimensional compact totally real minimal submanifold immersed in CP^n . We denote by K the sectional curvature of M . K. Ogiue [2] proved that if $K > (n-2)c/4(2n-1)$, then M is totally geodesic. This result was improved by B. Y. Chen and C. S. Houh [1] who obtained that if $K \geq (n-2)c/4(2n-1)$, then either M is totally geodesic or $n=2$ and M is a flat surface. This result for $n=2$ was proved by S. T. Yau [3]. In this paper, using methods different from those of [1] and [2], we prove

THEOREM. *Let M be an n -dimensional compact totally real minimal submanifold immersed in CP^n . If $K > 0$, then M is totally geodesic.*

2. Preliminaries. Let M be an n -dimensional totally real submanifold immersed in CP^n . For any vector fields X and Y tangent to M , we have

$$(1) \quad A_{JY}X = -J\sigma(X, Y), \quad D_X JY = J\nabla_X Y,$$

where A (respectively σ) denotes the Weingarten endomorphism (respectively, the second fundamental form) of M in CP^n , ∇ (respectively D) denotes the Riemannian connection of M (respectively the normal connection of M in CP^n), and J denotes the complex structure of CP^n .

From (1), and using the equations of Gauss and Ricci, it is easy to obtain

$$(2) \quad R^\perp(X, Y, JZ, JW) = R(X, Y, Z, W)$$

for any vector fields X, Y, Z and W tangent to M , where R and $R^\perp$ denote the curvature tensors associated with ∇ and D respectively.

If $\tilde{\nabla}\sigma$ denotes the covariant derivative of σ , we define the second covariant derivative of σ by

$$\begin{aligned} (\tilde{\nabla}^2\sigma)(X, Y, Z, W) &= D_X((\tilde{\nabla}\sigma)(Y, Z, W)) - (\tilde{\nabla}\sigma)(\nabla_X Y, Z, W) \\ &\quad - (\tilde{\nabla}\sigma)(Y, \nabla_X Z, W) - (\tilde{\nabla}\sigma)(Y, Z, \nabla_X W). \end{aligned}$$

Received by the editors March 9, 1984.

1980 *Mathematics Subject Classification*. Primary 53C40.

©1985 American Mathematical Society
0002-9939/85 \$1.00 + \$.25 per page

It is clear that $\tilde{\nabla}\sigma$ is symmetric, and

$$(3) \quad (\tilde{\nabla}^2\sigma)(X, Y, Z, W) = (\tilde{\nabla}^2\sigma)(Y, X, Z, W) + R^\perp(X, Y)\sigma(Z, W) \\ - \sigma(R(X, Y)Z, W) - \sigma(R(X, Y)W, Z).$$

3. Proof of Theorem. Let T_1M be the unit tangent bundle of M , and $f: T_1M \rightarrow \mathbf{R}$ the function given by $f(v) = \langle \sigma(v, v), Jv \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the metric of CP^n as well as that of M . Since T_1M is compact, f attains the maximum at a unit vector v tangent to M at a point p . For any unit vector u tangent to M at p , let $\alpha(t) = (\gamma(t), V(t))$, let $t \in (-\delta, \delta)$ be a curve in T_1M such that $\gamma(t)$ is the only geodesic in M with $\gamma(0) = p$ and $\gamma'(0) = u$, and let $V(t)$ be the parallel vector field along γ with $V(0) = v$. Then, using (1) we have

$$0 = df_v(u) = (d/dt)_{t=0} \langle \sigma(V(t), V(t)), JV(t) \rangle = \langle (\tilde{\nabla}\sigma)(u, v, v), Jv \rangle$$

and

$$(4) \quad 0 \geq d^2f_v(u, u) = \langle (\tilde{\nabla}^2\sigma)(u, u, v, v), Jv \rangle.$$

Now, using (1), (2) and (3), we have

$$(5) \quad d^2f_v(u, u) = \langle (\tilde{\nabla}^2\sigma)(v, v, u, u), Jv \rangle \\ + 2R(u, v, v, J\sigma(u, v)) + R(u, v, u, J\sigma(v, v)).$$

On the other hand, let S be the fiber of T_1M over p . Then f/S attains the maximum at v , and so, if $\beta(t)$, $t \in (-\delta, \delta)$ is a curve in S with $\beta(0) = v$, $\|\beta'(t)\| = 1$ and $\beta'(0) = u$, we have, using (1),

$$(6) \quad 0 = d(f/S)_v(u) = (d/dt)_{t=0} \langle \sigma(\beta(t), \beta(t)), J\beta(t) \rangle = 3\langle \sigma(v, v), Ju \rangle$$

and

$$(7) \quad 0 \geq d^2(f/S)_v(u, u) = 6\langle \sigma(\beta'(0), v), J\beta'(0) \rangle + 3\langle \sigma(v, v), J\beta''(0) \rangle \\ = 6\langle \sigma(u, v), Ju \rangle - 3\langle \sigma(v, v), Jv \rangle = 6\langle \sigma(u, u), Jv \rangle - 3f(v).$$

Since (6) is true for any unit vector u orthogonal to v , we have $\sigma(v, v) = f(v)Jv$, and so, from (1) v is an eigenvector of A_{Jv} with eigenvalue $f(v)$. Now, we can choose an orthonormal basis $\{u_1, \dots, u_n\}$ of T_pM which diagonalizes A_{Jv} such that $u_n = v$. If ρ_i , $1 \leq i \leq n-1$, are the eigenvalues of u_i , we have from (7)

$$(8) \quad f(v) - 2\rho_i \geq 0.$$

Now, from (1), (4), (5) and given that M is minimal, we have

$$0 \geq \sum_{i=1}^n d^2f_v(u_i, u_i) = \sum_{i=1}^{n-1} K(v, u_i) \{f(v) - 2\rho_i\}$$

and so from (8), given that $K > 0$, it follows that $\rho_i = f(v)/2$ for $1 \leq i \leq n-1$. Then $0 = \text{Trace } A_{Jv} = (n+1)f(v)/2$, which implies $f(v) = 0$. Since $f(-u) = -f(u)$, and v is a maximum for f , we have $f = 0$. Now, in the same way that we used to obtain (6), we have $\sigma = 0$.

ACKNOWLEDGEMENT. The author wishes to thank Professor A. Ros for his help.

REFERENCES

1. B. Y. Chen and C. S. Houh, *Totally real submanifolds of a quaternion projective space*, Ann. Mat. Pura Appl. (4) **120** (1979), 185–199.
2. K. Ogiue, *Positively curved totally real minimal submanifolds immersed in a complex projective space*, Proc. Amer. Math. Soc. **56** (1976), 264–266.
3. S. T. Yau, *Submanifolds with constant mean curvature. I*, Amer. J. Math. **96** (1974), 346–366.

DEPARTAMENTO DE GEOMETRIA Y TOPOLOGIA, FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA,
GRANADA, SPAIN