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A NONSTANDARD FUNCTIONAL APPROACH TO FUBINI'S THEOREM

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ABSTRACT. In this note we use a functional approach to the integral to obtain a special case of the Keisler-Fubini theorem; the general case can be obtained with a similar proof. An immediate application is the standard Fubini theorem for products of Radon measures. Similar methods give the Weil formula for quotient groups of compact Abelian groups.

1. Introduction. In this note we use a functional approach to the integral to obtain a special case of the Keisler-Fubini theorem[2] [**2**]; the general case can be obtained with a similar proof. An immediate application is the standard Fubini theorem for products of Radon measures, e.g., product Lebesgue measure. Similar methods give the Weil formula for quotient groups of compact Abelian groups. Throughout this note, $^*R$ denotes the set of nonstandard real numbers in a fixed enlargement of a structure containing the real numbers R. (See [**7**, **1**, **3** or **8**].) For any finite number $a \in {}^*R$, ${}^0a$ denotes the standard number nearest to a in R; we write $a \simeq 0$ if ${}^0a = 0$.

Let X be a compact Hausdorff space, and let $C(X)$ be the set of continuous real-valued functions on X. We let I_{*X} denote the nonstandard extension of a positive linear functional I_X on $C(X)$, e.g., the extension of the Riemann integral on a finite closed interval X in R^n. An $^*R$-valued function h on $^*X$ is called a null function if for any $\varepsilon > 0$ in R there is a function φ in the nonstandard extension $^*C(X)$ of $C(X)$ such that $|h| \leqslant \varphi$ and $I_{*X}(\varphi) < \varepsilon$. We let L_{*X} denote the class of real-valued functions f on $^*X$ such that $f = \varphi + h$, where $\varphi \in {}^*C(X)$ and h is null. Given such a decomposition of $f \in L_{*X}$, we set $J_{*X}(f) = {}^0I_{*X}(\varphi)$; if f is bounded, we may assume that h and φ are also bounded.

In [**4** and **5**] the author showed that L_{*X} is a vector lattice over R and J_{*X} is a well-defined positive linear functional on L_{*X} with the following monotone convergence property: If $\{f_n\colon n \in N\}$ is a monotone sequence in L_{*X} converging to a real-valued function F with $\sup_n |J_{*X}(f_n)| < +\infty$, then $F \in L_{*X}$ and $J_{*X}(F) = \lim_n J_{*X}(f_n)$. To obtain this property in [**4**], one needs an additional assumption ($\aleph_1$-saturation) which we assume here as well. It follows, as usual, from the

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[2] Proved in [**2**] for the hyperfinite case.

monotone convergence property in [**4**] that if $\mathfrak{M}$ is the collection of those subsets A of $^*X$ for which the characteristic function χ_A is in L_{*X} and $m(A) = J_{*X}(\chi_A)$ for each $A \in \mathfrak{M}$, then the triple $(^*X, \mathfrak{M}, m)$ is a standard complete measure space and the functional $J_{*X}(f)$ equals $\int f\,dm$ for each $f \in L_{*X}$.

If φ is a finite-valued function in $^*C(X)$ and $^0\varphi(x) = {}^0(\varphi(x))$ for each $x \in {}^*X$, then $^0\varphi - \varphi$ is null since $|^0\varphi(x) - \varphi(x)| \simeq 0$ for all $x \in {}^*X$. It follows that $^0\varphi = \varphi + (^0\varphi - \varphi) \in L_{*X}$ and $J_{*X}(^0\varphi) = {}^0I_{*X}(\varphi)$.

Each point $x \in {}^*X$ is in the monad (i.e., the intersection of the nonstandard extensions of all standard neighborhoods) of a unique point $z \in X$. We write $z = {}^0x$. If g is a real-valued function on the standard space X, then for each x in $^*X$ we set $\tilde{g}(x) = g(^0x)$ (i.e., the value of g at ^0x is spread out over the monad of ^0x). If $A \subseteq X$, then $\tilde{A}$ is the set in $^*X$ with characteristic function $\chi_{\tilde{A}} = \tilde{\chi}_A$. The author showed in [**4**] (working with continuous functions with compact support on locally compact spaces) that g is integrable with respect to the Radon measure μ representing I_X if and only if $\tilde{g} \in L_{*X}$, i.e., $\tilde{g} = \varphi + h$, where $\varphi \in {}^*C(X)$ and h is null. In this case, the integral of g, which we denote by $J_X(g)$, is equal to $J_{*X}(\tilde{g})$; indeed, μ can be obtained by setting $\mu(A) = m(\tilde{A})$ for each $A \subseteq X$ with $\chi_{\tilde{A}} \in L_{*X}$. If, for example, I_X is the Riemann integral on an interval X, then g is Lebesgue integrable on X if and only if $\tilde{g} \in L_{*X}$, and then the Lebesgue integral of g equals $J_{*X}(\tilde{g})$.

By a null set in X, we mean a measurable set B with Radon measure 0, i.e., $J_X(\chi_B) = 0$, and by a null set in $^*X$, we mean a set A with $m(A) = 0$. A property is said to hold for almost all points in X or $^*X$ if it holds except on a null set. If $f \geqslant 0$ is a real-valued function on $^*X$, then f is a null function if and only if $f \in L_{*X}$ and $J_{*X}(f) = 0$; of course, this holds if and only if $f(x) = 0$ for almost all $x \in {}^*X$ (see [**4**]).

Let us say that a function $f(u, v)$ on a product space $U \times V$ has the Fubini property with respect to integrals on U, V and $U \times V$ if for "almost all u", $f(u, \cdot)$ is integrable on V; for "almost all v", $f(\cdot, v)$ is integrable on U; and both iterated integrals exist and are equal to the integral of f over $U \times V$. The strong Fubini property is the Fubini property with "almost all u" and "almost all v" replaced by "all u" and "all v", respectively.

2. The Fubini theorems. Let X and Y be compact Hausdorff spaces, and let Z be the product space $X \times Y$. Let I_X and I_Y be positive linear functionals on $C(X)$ and $C(Y)$, respectively. By the Stone-Weierstrass theorem (applied to finite linear combinations of products of the form $g(x)h(y)$), the integrals $I_Y(f)$ and $I_X(f)$ are functions in $C(X)$ and $C(Y)$, respectively, for each $f \in C(Z)$; the iterated functionals I_XI_Y and I_YI_X are equal on $C(Z)$, so their common value is a positive linear functional I_Z with the strong Fubini property on $C(Z)$ (see [**6**, 16C]). It is this functional that one represents with a product measure μ, which, recall, one can obtain from J_{*Z}. By the transfer principle of nonstandard analysis, the "internal" strong Fubini property holds for the extension of the iterated functional, I_{*Z}, on $^*C(Z)$. For example, if X and Y are finite closed intervals in R^n and R^m, $1 \leqslant n, m$, and Z is the interval $X \times Y$, then for each $\varphi \in {}^*C(Z)$, the internal strong Fubini property holds with respect to the nonstandard extensions of the Riemann integrals on X, Y and Z.

We now use the internal strong Fubini property to prove a special case (Theorem 1) of H. J. Keisler's Fubini theorem **[2]**; the general case can be established with a similar proof using characteristic functions of measurable rectangles instead of continuous functions.

LEMMA 1. *If φ is a finite-valued function in $^*C(Z)$, then $^0\varphi$ has the strong Fubini property with respect to J_{*X}, J_{*Y}, and J_{*Z}.*

PROOF. For each $x \in {}^*X$, $\varphi(x,\cdot\,) \in {}^*C(Y)$, and $I_{*Y}(\varphi) \in {}^*C(X)$, so $^0\varphi(x,\cdot\,) \in L_{*Y}$ and $^0I_{*Y}(\varphi) \in L_{*X}$. Moreover,

$$J_{*Z}(^0\varphi) = {}^0I_{*Z}(\varphi) = {}^0I_{*X}I_{*Y}(\varphi) = J_{*X}{}^0I_{*Y}(\varphi) = J_{*X}J_{*Y}(^0\varphi).$$

A similar argument with the roles of X and Y reversed yields the result.

LEMMA 2. *If h is a bounded real-valued null function on $^*Z$, then h has the Fubini property with respect to J_{*X}, J_{*Y}, and J_{*Z}.*

PROOF. We may assume that $0 \leqslant h \leqslant K$ for some integer K. Since h is null, there is a decreasing sequence $\{\varphi_n\colon n \in N\} \subset {}^*C(Z)$ with $h \leqslant \varphi_n \leqslant K$ for all n and $\lim_n {}^0I_{*Z}(\varphi_n) = 0$. Since h is real-valued, there is a real-valued $H \in L_{*Z}$ with $^0\varphi_n \searrow H \geqslant h$. By Lemma 1 and the monotone convergence property, H has the strong Fubini property and is null. It follows that for almost all $x \in {}^*X$, $H(x,\cdot\,)$ is a null function on $^*Y$, and so $h(x,\cdot\,)$ is a null function on $^*Y$. Similarly, for almost all $y \in {}^*Y$, $h(\cdot\,, y)$ is a null function on $^*X$. Thus the Fubini property holds for h.

THEOREM 1. *If $f \in L_{*Z}$, then f has the Fubini property with respect to J_{*X}, J_{*Y}, and J_{*Z}.*

PROOF. By the monotone convergence property, we may assume that f is bounded. Let $f = \varphi + h$ be a decomposition of f with φ a bounded function in $^*C(Z)$ and h a bounded null function. Then $f = {}^0\varphi + (\varphi - {}^0\varphi) + h$. Since the null function $(\varphi - {}^0\varphi) + h$ is real valued, the theorem follows from Lemmas 1 and 2.

To prove the standard Fubini theorem for the product of Radon measures on locally compact spaces, it is sufficient, by regularity of measures and the monotone convergence theorem, to establish the following result. The Fubini theorem for the product of Lebesgue measures is an immediate corollary.

THEOREM 2. *Let X and Y be compact Hausdorff spaces, and let $Z = X \times Y$. Let I_X and I_Y be positive linear functionals on $C(X)$ and $C(Y)$, respectively, and let I_Z be the iterated functional on $C(Z)$. If f is bounded and integrable with respect to J_Z, then f has the Fubini property with respect to the integrals J_X, J_Y, and J_Z.*

PROOF. By Theorem 1, the Fubini property holds for $\tilde{f}(x, y)$ on $^*Z$. If $^0x_1 = {}^0x_2$, then for all $y \in {}^*Y$, $\tilde{f}(x_1, y) = \tilde{f}(x_2, y)$. Thus there is a standard set $A \subset X$ such that for all $x \in {}^*X - \tilde{A}$, $\tilde{f}(x,\cdot\,) \in L_{*Y}$, and $\tilde{A}$ is null in $^*X$ so A is null in X. If

$x \in X - A$, then $\widetilde{f(x,\cdot)} = \tilde{f}(x,\cdot)$ on ${}^*Y$, so $J_Y f(x,\cdot) = J_{*Y}\tilde{f}(x,\cdot)$. Set $J_Y f(x,\cdot) = 0$ for $x \in A$. Since $\widetilde{J_Y f(x,\cdot)} = J_{*Y}\tilde{f}(x,\cdot)$ on ${}^*X - \tilde{A}$, $J_Y f(x, y)$ is integrable on X, and

$$J_X J_Y f(x, y) = J_{*X} J_{*Y}\tilde{f}(x, y) = J_{*Z}\tilde{f}(x, y) = J_Z f(x, y).$$

A similar argument with the roles of X and Y reversed gives the result.

Note that this functional approach to Fubini's theorem works not just for Baire functions, but for all functions integrable with respect to the completion of the representing Radon measure on the Borel sets of Z. Earl Berkson has pointed out to the author that essentially the same methods as those used above will give the Weil formula at least for quotient groups of compact Abelian groups. Indeed, the formula stated for Baire functions in Loomis [**6**, Corollary, p. 131] holds for all functions integrable with respect to the completion of Haar measure on the Borel sets.

References

1. A. E. Hurd and P. A. Loeb, *An introduction to nonstandard real analysis*, Academic Press (to appear).

2. H. J. Keisler, *An infinitesimal approach to stochastic analysis*, Mem. Amer. Math. Soc. No. 297 (1984).

3. P. A. Loeb, *An introduction to nonstandard analysis and hyperfinite probability theory*, Probabilistic Analysis and Related Topics (A. T. Bharucha-Reid, ed.), Academic Press, New York, 1979, pp. 105–142.

4. ______, *A functional approach to nonstandard measure theory*, Proc. S. Kakutani Retirement Conf., Contemporary Math., vol. 26, Amer. Math. Soc., Providence, R. I., 1984, pp. 251–261.

5. ______, *Measure spaces in nonstandard models underlying standard stochastic processes*. Proc. Internat. Congr. of Math. (Warsaw, 1983) (to appear).

6. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, Princeton, N. J., 1953.

7. A. Robinson, *Non-standard analysis*, North-Holland, Amsterdam, 1966.

8. K. D. Stroyan and W. A. J. Luxemburg, *Introduction to the theory of infinitesimals*, Pure and Appl. Math., Vol. 72, Academic Press, New York, 1976.

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