A SPURIOUS BROWNIAN MOTION

CLYDE D. HARDIN, JR.

ABSTRACT. There exists an $\mathbf{R}^d$ -valued mean-zero Gaussian process, all of whose projections agree with the projections of standard Brownian motion, yet which is not standard Brownian motion.

Suppose that $\{W(t) = (W_1(t), \dots, W_d(t)): t \geq 0\}$ is an $\mathbf{R}^d$ -valued mean-zero Gaussian process such that all projections $Y(t) = \sum_{j=1}^d \lambda_j W_j(t)$ behave as if $\{W(t)\}$ were standard Brownian motion in $\mathbf{R}^d$, i.e. $EY(t)Y(s) = (\sum \lambda_j^2) \min(t, s)$. Is $\{W(t)\}$ in fact standard Brownian motion in $\mathbf{R}^d$?

Questions of this sort arise naturally when one tries to prove multidimensional analogs of some one-dimensional results by using projection methods (see, e.g., [1]).

Clearly each component of $\{W(t)\}$ is standard (one-dimensional) Brownian motion. Also, one can show that under certain "reasonable" additional hypotheses (if $\{W(t)\}$ is a martingale, for instance) that the answer to our question is yes. The following example shows that in general the (perhaps surprising) answer is no.

Let $B_1(t)$ and $B_2(t)$ be two independent standard Brownian motions. Set

$$W_1(t) = B_1(2t/3) - B_2(t/3) \quad \text{and} \quad W_2(t) = B_1(t/3) + B_2(2t/3).$$

By checking covariances the reader may verify that $W(t) = (W_1(t), W_2(t))$ has the desired properties. Yet, $\{W_1(t)\}$ and $\{W_2(t)\}$ are not independent processes, so $\{W(t)\}$ is not standard Brownian motion in $\mathbf{R}^2$.

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REFERENCES

1. T. G. Kurtz, *The central limit theorem for Markov chains*, Ann. Probab. **9** (1981), 557–560.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MILWAUKEE, WISCONSIN 53201

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