CONTINUOUS SELECTIONS FOR WEAKLY HAUSDORFF LOWER SEMICONTINUOUS MULTIFUNCTIONS

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ABSTRACT. The existence of continuous selections is proved for a class of weakly Hausdorff lower semicontinuous multifunctions.

1. Let X be a topological space and let Y be a normed space with norm $|\cdot|$. Let 2^Y be the collection of all nonempty subsets of Y . Set $\mathcal{C}(Y) = \{A \in 2^Y \mid A \text{ is convex}\}$ and $\mathcal{C}_c(Y) = \{A \in 2^Y \mid A \text{ is convex closed}\}$. For $a \in Y$ and $B \subset Y$ ($B \neq \emptyset$) we put $d(a, B) = \inf\{|a - b| \mid b \in B\}$. By $S(u, r)$ we denote the open ball in Y of radius $r > 0$ and center u . We set $S = S(0, 1)$. By a neighborhood of a point $x \in X$ we always mean a set open in X containing x .

Let $F: X \rightarrow 2^Y$. Let $\varepsilon > 0$. A function $f: X \rightarrow Y$ satisfying $f(x) \in F(x)$ (resp., $f(x) \in F(x) + \varepsilon S$) for every $x \in X$ is called a selection (resp., ε -selection) of F .

A multifunction $F: X \rightarrow 2^Y$ is called almost lower semicontinuous (a-l.s.c.)—see [2]—iff for every $x_0 \in X$ and for every $\varepsilon > 0$ there is a neighborhood U of x_0 such that $\bigcap\{F(x) + \varepsilon S \mid x \in U\} \neq \emptyset$.

In [2] it is proved that a multifunction $F: X \rightarrow \mathcal{C}(Y)$ is a-l.s.c. iff F admits continuous ε -selections for each $\varepsilon > 0$. The example below shows that there exist multifunctions with nonempty convex closed values which are a-l.s.c. and do not admit continuous selections. Another example of this type is also contained in [2].

EXAMPLE. Let c_0 be the Banach space of all real sequences $y = (y_1, y_2, \dots)$ such that $\lim_{n \rightarrow \infty} y_n = 0$, with norm $|y| = \sup\{|y_n| \mid n \in \mathbb{N}\}$. Let $A \subset c_0$ be a closed bounded symmetric convex body having the following property: for each $y \in c_0 \setminus A$ there is no point $a_0 \in A$ such that $|y - a_0| = d(y, A)$. The existence of such a set has been established by Edelstein and Thompson [3]. Now, fix $y_0 \in c_0 \setminus A$ and define the multifunction $F: \mathbb{R} \rightarrow \mathcal{C}_c(c_0)$ by $F(x) = A$ if $x \leq 0$, and $F(x) = \{y \in c_0 \mid |y - y_0| \leq d(y_0, A)\}$ if $x > 0$. Clearly F is a-l.s.c. Suppose F has a continuous selection f . For $x > 0$ we have $|f(x) - y_0| \leq d(y_0, A)$; hence, by continuity, $|f(0) - y_0| \leq d(y_0, A)$. But $f(0) \in A$, so $|f(0) - y_0| = d(y_0, A)$, a contradiction. It follows that F does not admit continuous selections.

A multifunction $F: X \rightarrow 2^Y$ is called weakly Hausdorff lower semicontinuous (H_w -l.s.c.) at $x_0 \in X$ iff, for every $\varepsilon > 0$ and every neighborhood V of x_0 there are a neighborhood U of x_0 ($U \subset V$) and a point $x' \in U$ such that $F(x') \subset F(x) + \varepsilon S$ for every $x \in U$. If F is H_w -l.s.c. at each $x_0 \in X$, then F is called H_w -l.s.c.

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If $x' = x_0$ we obtain the definition of a Hausdorff lower semicontinuous (H-l.s.c.) multifunction at x_0 . Evidently, each H-l.s.c. multifunction is also H_w -l.s.c., and each H_w -l.s.c. multifunction is also a-l.s.c. The converses are not true in general.

THEOREM. *Let X be a paracompact space and let Y be a Banach space. Then every H_w -l.s.c. multifunction $F: X \rightarrow \mathcal{C}_c(Y)$ admits a continuous selection.*

Since a multifunction $F: X \rightarrow 2^Y$ which is l.s.c. (that is, for each set V open in Y the set $\{x \in X \mid F(x) \cap V \neq \emptyset\}$ is open in X) is not necessarily H_w -l.s.c. (and vice versa), the above result does not follow from Michael's selection theorem [4]. A particular case of the above Theorem can be found in [1].

REMARK. Consider the Cauchy problem

$$(*) \quad x' \in F(x), \quad x(0) = x_0 \quad \text{where } F: \mathbf{R}^n \rightarrow \mathcal{C}_c(\mathbf{R}^n) \text{ is } H_w\text{-l.s.c. and } x_0 \in \mathbf{R}^n.$$

From the above selection result and Peano's theorem, it follows that $(*)$ has (local) solutions.

2. LEMMA. *Let X be a topological space and let Y be a normed space. Let $F: X \rightarrow \mathcal{C}(Y)$ and $f: X \rightarrow Y$ satisfy $F(x) \cap S(f(x), r/2) \neq \emptyset$ ($r > 0$) for each $x \in X$. Then for every $\varepsilon > 0$ there exists a $\sigma = \sigma(\varepsilon, r) > 0$ (independent of x) such that*

$$(1) \quad [F(x) + \sigma S] \cap [S(f(x), r) + \sigma S] \subset F(x) \cap S(f(x), r) + \varepsilon S$$

for each $x \in X$.

PROOF. Let $\varepsilon > 0$. Let σ satisfy $0 < \sigma < \min\{\varepsilon/15, r/4\}$. We prove that with such a choice of σ , (1) is fulfilled. To this end fix $x \in X$ and let $y \in [F(x) + \sigma S] \cap [S(f(x), r) + \sigma S]$ be arbitrary.

Case 1. Suppose $|y - f(x)| \leq r - \sigma$. Since $y \in F(x) + \sigma S$, there is $u \in F(x)$ such that $|u - y| < \sigma$. Moreover,

$$|u - f(x)| \leq |u - y| + |y - f(x)| < \sigma + (r - \sigma),$$

that is, $u \in S(f(x), r)$. Consequently, since $|y - u| < \sigma < \varepsilon$, we have

$$y = u + (y - u) \in F(x) \cap S(f(x), r) + \varepsilon S.$$

Case 2. Suppose $|y - f(x)| > r - \sigma$. Consider the family of open balls $\Sigma_\lambda = S(u_\lambda, r_\lambda)$, $0 < \lambda < 1$, where

$$u_\lambda = (1 - \lambda)y + \lambda f(x), \quad r_\lambda = r\lambda/2.$$

Let $0 < \tilde{\lambda} < 1$ be such that $|u_{\tilde{\lambda}} - f(x)| = r - \sigma - r_{\tilde{\lambda}}$. An easy computation shows that the required $\tilde{\lambda}$ is given by

$$\tilde{\lambda} = \frac{|y - f(x)| - r + \sigma}{|y - f(x)| - r/2}.$$

Evidently, $r - \sigma < |y - f(x)| < r + \sigma$; thus,

$$0 < \tilde{\lambda} < \frac{(r + \sigma) - r + \sigma}{(r - \sigma) - r/2} = \frac{2\sigma}{r/2 - \sigma} < \frac{8\sigma}{r}.$$

Observe that $r_{\tilde{\lambda}} < 4\sigma$, and $|u_{\tilde{\lambda}} - y| < 10\sigma$ because

$$|u_{\tilde{\lambda}} - y| = \tilde{\lambda}|f(x) - y| < (8\sigma/r)(r + \sigma) < 10\sigma.$$

Moreover,

$$(2) \quad \Sigma_{\tilde{\lambda}} \subset S(f(x), r - \sigma)$$

because each $u \in \Sigma_{\tilde{\lambda}}$ satisfies

$$|u - f(x)| \leq |u - u_{\tilde{\lambda}}| + |u_{\tilde{\lambda}} - f(x)| < r_{\tilde{\lambda}} + (r - \sigma - r_{\tilde{\lambda}}) = r - \sigma.$$

Now, fix a point z in the set $F(x) \cap S(f(x), r/2)$. Let $v_{\tilde{\lambda}} = (1 - \tilde{\lambda})y + \tilde{\lambda}z$. Clearly, $v_{\tilde{\lambda}} \in F(x) + \sigma S$, thus, there is a $w \in F(x)$ such that $|w - v_{\tilde{\lambda}}| < \sigma$. Furthermore, $|v_{\tilde{\lambda}} - u_{\tilde{\lambda}}| = \tilde{\lambda}|z - f(x)| < \tilde{\lambda}r/2 = r_{\tilde{\lambda}}$, that is, $v_{\tilde{\lambda}} \in \Sigma_{\tilde{\lambda}}$. From this and (2) we have $|v_{\tilde{\lambda}} - f(x)| < r - \sigma$. Whence,

$$|w - f(x)| \leq |w - v_{\tilde{\lambda}}| + |v_{\tilde{\lambda}} - f(x)| < \sigma + (r - \sigma) = r,$$

that is, $w \in S(f(x), r)$. Consequently, $w \in F(x) \cap S(f(x), r)$.

On the other hand,

$$\begin{aligned} |w - y| &\leq |w - v_{\tilde{\lambda}}| + |v_{\tilde{\lambda}} - u_{\tilde{\lambda}}| + |u_{\tilde{\lambda}} - y| \\ &< \sigma + r_{\tilde{\lambda}} + 10\sigma < \sigma + 4\sigma + 10\sigma = 15\sigma, \end{aligned}$$

thus, $|w - y| < \varepsilon$ since $\sigma < \varepsilon/15$. Then

$$y = w + (y - w) \in F(x) \cap S(f(x), r) + \varepsilon S,$$

which completes the proof.

3. Proof of the Theorem. Let $\varepsilon_k = 2^{-k}$. Following Michael [4] we need only construct a sequence of continuous functions $f_k : X \rightarrow Y$ satisfying

$$(3) \quad d(f_k(x), F(x)) < \varepsilon_k, \quad k = 1, 2, \dots,$$

$$(4) \quad |f_k(x) - f_{k-1}(x)| < \varepsilon_{k-3}, \quad k = 2, 3, \dots,$$

for each $x \in X$. Indeed, if (3) and (4) are fulfilled, $\{f_k\}$ is uniformly Cauchy, so it converges to a continuous function which is a selection of F . We construct $\{f_k\}$ by induction. The existence of a continuous function f_1 satisfying (3) follows from [2, Theorem 2.4]. Suppose f_n ($n \geq 1$) is a continuous function satisfying (3).

Define $\psi_n : X \rightarrow \mathcal{C}(Y)$ by

$$\psi_n(x) = F(x) \cap S(f_n(x), \varepsilon_{n-1}).$$

We claim that ψ_n is H_w -l.s.c. To this end fix $x_0 \in X$. Let $\varepsilon > 0$ and let V be any neighborhood of x_0 . By the Lemma (with $F, f_n, \varepsilon_{n-1}$ in the place of F, f, r) there is a $\sigma = \sigma(\varepsilon, \varepsilon_{n-1}) > 0$ such that

$$(5) \quad [F(x) + \sigma S] \cap [S(f_n(x), \varepsilon_{n-1}) + \sigma S] \subset F(x) \cap S(f_n(x), \varepsilon_{n-1}) + \varepsilon S$$

for each $x \in X$. By the continuity of f_n at x_0 there is a neighborhood $\tilde{V}$ ($\tilde{V} \subset V$) of x_0 such that

$$(6) \quad S(f_n(x_1), \varepsilon_{n-1}) \subset S(f_n(x_2), \varepsilon_{n-1}) + \sigma S \quad \text{for each } x_1, x_2 \in \tilde{V}.$$

Since F is H_w -l.s.c. at x_0 , there are a neighborhood U of x_0 ($U \subset \tilde{V}$) and a point $x' \in U$ such that

$$(7) \quad F(x') \subset F(x) + \sigma S \quad \text{for each } x \in U.$$

By virtue of (7), (6) and (5) we have

$$\begin{aligned} F(x') \cap S(f_n(x'), \varepsilon_{n-1}) &\subset [F(x) + \sigma S] \cap [S(f_n(x), \varepsilon_{n-1}) + \sigma S] \\ &\subset F(x) \cap S(f_n(x), \varepsilon_{n-1}) + \varepsilon S \end{aligned}$$

for each $x \in U$. Hence ψ_n is H_w -l.s.c. By [2, Theorem 2.4] there is a continuous function $f_{n+1} : X \rightarrow Y$ such that $d(f_{n+1}(x), \psi_n(x)) < \varepsilon_{n+1}$, $x \in X$. Clearly f_{n+1} satisfies (3) and (4) (with $n+1$ in the place of k). This completes the proof.

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