MAXIMAL COMPACT NORMAL SUBGROUPS AND PRO-LIE GROUPS

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ABSTRACT. We are concerned with conditions under which a locally compact group G has a maximal compact normal subgroup K and whether or not G/K is a Lie group. If G has small compact normal subgroups K such that G/K is a Lie group, then G is pro-Lie. If in G there is a collection of closed normal subgroups $\{H_\alpha\}$ such that $\bigcap H_\alpha = e$ and G/H_α is a Lie group for each α , then G is a residual Lie group. We determine conditions under which a residual Lie group is pro-Lie and give an example of a residual Lie group which is not embeddable in a pro-Lie group.

DEFINITION 1. A locally compact topological group G is an L -group if, for each neighborhood U of the identity and compact subset C of G , there is a neighborhood V such that $gHg^{-1} \cap C \subset U$ for every $g \in G$ and subgroup $H \subset V$.

DEFINITION 2. A topological group G is *pro-Lie* if G has small compact normal subgroups K such that G/K is a Lie group.

DEFINITION 3. A locally compact group G is a *residual Lie group* if there is a collection of closed normal subgroups $\{H_\alpha\}$ such that $\bigcap H_\alpha = e$ and G/H_α is a Lie group for each α .

It is easy to see that a residual Lie group is an L -group [1, Theorem 1.3 and its Corollary]. In this paper we obtain the following: If G is an L -group and has an open normal subgroup G_1 such that G_1/G_0 is compact, then G is pro-Lie. There exists a locally compact group G with a maximal compact normal subgroup K such that G/K is not a Lie group. A residual Lie group is not necessarily embeddable by a continuous isomorphism in a pro-Lie group. A generalized FC-group has a maximal compact normal subgroup.

PROPOSITION 1. *For a locally compact group G , the following are equivalent:*

- (1) G has a compact normal subgroup K such that G/K is a Lie group.
- (2) G has a compact normal subgroup K such that G/K is locally connected.
- (3) G has an open normal subgroup G_1 such that G_1/G_0 is compact.

We denote the connected component of the identity of G by G_0 .

PROOF. Obviously (1) implies (2). We assume that K is compact normal and G/K is locally connected. Thus $(G/K)_0 = G_0/K$ is open in G/K . It follows that $KG_0 = G_1$ is an open normal subgroup of G and G_1/G_0 is compact. We complete the proof by showing that (3) implies (1). Assuming (3) we have a maximal compact normal subgroup K of G_1 and G_1/K is a Lie group [1, Proposition 2.6]. Since K is maximal in G_1 and G_1 is normal in G , K is normal in G . Since G/G_1 is discrete and G_1/K is a Lie group, G/K is a Lie group.

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THEOREM 2. *For a locally compact group G , the following are equivalent:*

- (1) G is pro-Lie.
- (2) G is an L-group and G/G_0 is pro-Lie.
- (3) G is an L-group and has a compact normal subgroup K such that G/K is locally connected.
- (4) G is an L-group and has an open normal subgroup G_1 such that G_1/G_0 is compact.

PROOF. Obviously (1) implies (3). The reverse implication follows from Proposition 1 and [1, Theorem 1.2 and Corollary 2]. Also by Proposition 1, (3) and (4) are equivalent. Since (1) obviously implies (2) we can complete the proof by showing that (2) implies (4). This follows immediately from [1, Proposition 1.7 and Corollary 2].

If G is a locally compact group and K is a maximal compact normal subgroup, then there are some obvious cases when G/K is a Lie group. For example:

- (1) If there is an open normal subgroup G_1 such that G_1/G_0 is compact, then G/K is a Lie group. Now G_1 has a maximal compact normal subgroup K_1 such that G_1/K_1 is a Lie group. Also K_1 is normal in G ; thus, G/K_1 is a Lie group. It follows that, if K is maximal compact normal in G , then G/K is a Lie group.
- (2) If G/H is locally connected for some compact normal subgroup H , then G/K is a Lie group.
- (3) If G/K is pro-Lie, then it is a Lie group.

By the semidirect product $H \times_{\eta} K$ of two groups H and K we mean the group determined by a homomorphism $\eta: K \rightarrow \mathcal{A}(H)$, the automorphism group of H , with the group operation in $H \times_{\eta} K$ defined by $(h, k)(h_1, k_1) = (h\eta(k)(h_1), kk_1)$.

EXAMPLE 1. Let $G = (\sum_{i=-1}^{\infty} K_i) \times (\prod_{i=1}^{\infty} K_i) \times_{\eta} Z$, where $K_i = \{0, 1\}$ and the integer $n \in Z$ shifts coordinates n places to the left. We assume $\sum K_i$ has the discrete topology and $\prod K_i$ has the product topology. It is easy to see that any nontrivial invariant subset C of G contains elements $(k(n), z^j)$, where the n th coordinate of $k(n)$ is 1 for infinitely many integers n , and j is some fixed integer. Thus C is not compact. It follows that the maximal compact normal subgroup of G is the identity. Thus we have a compactly generated group with maximal compact normal subgroup K such that G/K is not a Lie group.

There are many examples of topological groups without maximal compact normal subgroups; for example, any discrete group with no maximal finite subgroup.

EXAMPLE 2. We give another example of a locally compact group G which is not a Lie group and in which the identity is the maximal compact normal subgroup. Let $G = \sum_{i=1}^{\infty} H_i \times_{\eta} \prod_{i=0}^{\infty} K_i$, where $H_i = Z$, the additive group of integers, and $K_i = Z_2$, the multiplicative two-element group. The automorphism $\eta(k)$ of $\sum H_i$ is defined as $\eta(k)(h)_i = k_i h_i$. It is easy to see that if $\sum H_i$ has the discrete topology and $\prod K_i$ the product topology, G has the desired properties.

DEFINITION. A locally compact group G is a generalized FC-group if $G = A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} = e$, where each A_i is a closed normal subgroup of G and A_i/A_{i+1} is a compactly generated FC-group.

We note that by [2, Corollary 3.9], every compactly generated FC-group has a compact normal subgroup such that the corresponding factor group is of the form $V \times Z^k$.

THEOREM 3. *If G is a locally compact generalized FC-group, then G has a maximal compact normal subgroup.*

PROOF. We proceed by induction, noting that, for $n = 1$, the conclusion follows from the corollary referred to above. Let $G = A_1 \supset A_2 \supset \cdots \supset A_{n+1} = e$, where each A_i/A_{i+1} is a compactly generated FC-group. We assume that every generalized FC-group with sequence $\{A_i\}$ of shorter length has a maximal compact normal subgroup. Thus G/A_n has a maximal compact normal subgroup M and A_n has a maximal compact normal subgroup K . Let $N = \pi^{-1}(M)$, where π is the canonical mapping of G onto G/A_n . Again, using [2, Corollary 3.9], we let $Q = Z(A_n/K, N/K)$ and note that Q is a compactly generated central group. Thus Q has a maximal compact normal subgroup. We let P be the inverse image of this subgroup under the canonical mapping of N onto N/K . We show that P is the maximal compact normal subgroup of G . Let L be any compact normal subgroup of G . Then $\pi(L) \subset M$ and $L \subset N$. Also $L/K \cap A_n/K = \{e\}$ since $A_n/K \cong V \times Z^k$. It follows that $L/K \subset Q$ and $L \subset P$ as desired.

DEFINITION 4. A topological group G is an N -group if for every neighborhood U of the identity and compact set C of G , there is a neighborhood V of the identity such that $gVg^{-1} \subset U \cup (G - C)$ for all $g \in G$. A group G is a SIN -group if G has small neighborhoods of the identity which are invariant under the inner automorphisms of G .

It is easy to see that, if a topological group G can be embedded in an SIN -group, then G is an N -group.

EXAMPLE 3. We construct an example of an N -group which is residual Lie and which cannot be embedded in a pro-Lie group. We begin with the group F of 2×2 matrices of determinant equal to 1. Let H be the subgroup of F of matrices of the form $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$. If N_0 is a normal subgroup of $F \times F$ which contains all elements of the form $[A, A^{-1}]$, then N_0 contains all pairs of the form $[I, C]$, where I is the identity and C is an element of H . To show this, let $A = \begin{pmatrix} a/y & 1 \\ 0 & y/a \end{pmatrix}$ and $B = \begin{pmatrix} 1/a & 0 \\ 0 & a \end{pmatrix}$. Then

$$[AB, A^{-1}B^{-1}] = \left[\begin{pmatrix} 1/y & a \\ 0 & y \end{pmatrix}, \begin{pmatrix} y & -1/a \\ 0 & 1/y \end{pmatrix} \right]$$

is in N_0 . Thus, if

$$A = \begin{pmatrix} a^2 - 1/ca & a \\ 0 & ca/a^2 - 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} ca/a^2 - 1 & -1/a \\ 0 & a^2 - 1/ca \end{pmatrix},$$

then $[A, B]$ is in N_0 . It follows that $[I, \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}]$ is in N_0 as desired, since $[I, \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}] = [A, B][A^{-1}, A]$.

If $c > 0$ and $C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, then $[I, C]$ is the conjugate of $[I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}]$ by $[A^{-1}, A]$, where

$$A = \begin{pmatrix} \sqrt{c} & 1 \\ 0 & 1/\sqrt{c} \end{pmatrix}.$$

If $c < 0$, then $[I, C]$ is the conjugate of $[I, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}]$ by $[A^{-1}, A]$, where

$$A = \begin{pmatrix} \sqrt{-c} & -1 \\ 0 & 1/\sqrt{-c} \end{pmatrix}.$$

Thus, $N_1 = I \times H$ has only three conjugacy classes relative to N_0 . We show that N_0 cannot be embedded (isomorphically) in a compact group.

LEMMA. *If G is a discrete group and H is an infinite subgroup which has only finitely many conjugacy classes relative to G , then G cannot be embedded in a compact group.*

PROOF. Suppose there exists an isomorphism $Q: G \rightarrow K$, a compact group. If H_0 is a conjugacy class of H such that $H_0 \neq \{e\}$, then there is a neighborhood V of the identity of K such that $Q(H_0) \cap V = \emptyset$. Because, if $Q(H_0)$ intersects every neighborhood of the identity of K , then for any neighborhood W there is a neighborhood V of the identity of K such that $gVg^{-1} \subset W$. If $Q(H_0) \cap V \neq \emptyset$, then $Q(H_0) \subset W$. Consequently $Q(H_0) = e$ which is a contradiction. Since H has only finitely many conjugacy classes, there is a neighborhood U of the identity of K such that $Q(H) \cap U = e$. Thus $Q(H)$ is discrete which contradicts the fact that K is compact.

We are ready to construct the example referred to above. Let $F_i = F \times F$, where F is the group of 2×2 matrices referred to above and $K_i = Z_2$ for each positive integer i . Let $G = \sum F_i \times_\eta \prod K_i$, where $\sum F_i$ is the direct sum with discrete topology, $\prod K_i$ is the direct product with product topology, and η is defined as follows: If the i th coordinate of $k \in \prod K_i$ is 1, then the i th coordinate of $\eta(k)(f)$ is (f_2, f_1) , where the i th coordinate of f is (f_1, f_2) . If the i th coordinate of k is 0, then, of course, $\eta(k)$ does not alter the i th coordinate of $f \in \sum F_i$.

We note that any open normal subgroup of G contains an isomorphic image of a subgroup N_0 of $F \times F$ such that $[A, A^{-1}] \in N_0$ for each $A \in F$. To see this we note that, if $k \in \prod K_i$ has i th coordinate 1 and all other coordinates 0, and $f \in \sum F_i$ has i th coordinate $[A, I]$, and all other coordinates $[I, I]$, then $(f, 0) \in \sum F_i \times_\eta \prod K_i$ and $(f, 0)(I, k)(f, 0)^{-1} = (f', k)$, where f' has i th coordinate $[A, A^{-1}]$ and all other coordinates $[I, I]$, where I here is the identity of F .

To show that G cannot be embedded in a pro-Lie group, assume there exists a continuous isomorphism $Q: F \rightarrow G_1$ and G_1 is pro-Lie. Let K be a normal compact subgroup of G_1 such that G_1/K is a Lie group. Then $G/Q^{-1}(K)$ is a Lie group, hence discrete since G is totally disconnected. It follows that $Q^{-1}(K)$ is an open normal subgroup of G . This contradicts the Lemma. Thus, the existence of the continuous isomorphism Q is contradicted.

Since a locally compact SIN-group is pro-Lie [2, Theorem 2.11], it follows from the above argument that G cannot be embedded in a locally compact SIN-group.

In conclusion we see that G is an N -group and residual Lie. Actually G can be embedded in $\sum F_i \times_\eta \prod K_i$, where $\sum F_i \times \prod K_i$ has the relative product topology of $\prod F_i \times \prod K_i$ which is SIN but, of course, not locally compact.

REFERENCES

1. R. W. Bagley, T. S. Wu and J. S. Yang, *Pro-Lie groups*, Trans. Amer. Math. Soc. (to appear).
2. S. Grosser and M. Moskowitz, *Compactness conditions in topological groups*, J. Reine Angew. Math. **246** (1971), 1-40.

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