A MODULUS OF CONTINUITY FOR A CLASS OF QUASI-SMOOTH FUNCTIONS

J. ERNEST WILKINS, JR.

ABSTRACT. Let Z be the class of real-valued functions $f(x)$, defined and continuous on the closed interval $I = [-1, 1]$, such that $f(-1) = f(1) = 0$ and

$$|f(\xi) - 2f\{(\xi + \eta)/2\} + f(\eta)| \leq |\xi - \eta|$$

for all ξ and η in I . We show that $\omega(h) = h \log_2\{2eK/(h \log_2 e)\}$ is a modulus of continuity on Z , if $K = \sup_{f \in Z} \max_{x \in I} |f(x)|$.

1. Introduction. Timan [3, 4, 5] calls a function $f(x)$, real-valued and continuous on a closed interval $[\alpha, \beta]$, quasi-smooth if it satisfies for some constant M the condition of Zygmund [7] that

$$(1) \quad |f(\xi) - 2f\{(\xi + \eta)/2\} + f(\eta)| \leq M|\xi - \eta|$$

for all ξ and η in $[\alpha, \beta]$. Let Z be the class of quasi-smooth functions for which $M = 1$, $(\alpha, \beta) = (-1, 1)$ and $f(-1) = f(1) = 0$. Suppose that

$$(2) \quad K = \sup_{f \in Z} \max_{x \in I} |f(x)|.$$

Timan [5] proved that $K \leq 4/3$, and that

$$(3) \quad \omega_0(h) = h\{(8/3) + 2K + \log_2(1/h)\}$$

is a uniform modulus of continuity on Z , i.e., that

$$|f(x) - f(y)| \leq \omega(|x - y|)$$

for all f in Z and all x and y in I , if $\omega = \omega_0$.

We will prove the following

THEOREM 1. *The function*

$$(4) \quad \omega(h) = h \log_2\{2eK/(h \log_2 e)\}$$

is a uniform modulus of continuity on Z .

Because it is known [6] that $13/10 \leq K \leq 1014/779 < 1.301669$, the constant $2eK/\log_2 e$ in (4) does not exceed 4.905130, whereas the corresponding constant $2^{(8/3)+2K}$ in the Timan result (3) is at least $2^{79/15} > 38.49680$.

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Timan [5] introduced the function $\theta(x)$ defined so that

$$(5) \quad \sup|f(x)| = (1 - |x|) \left[\log_2 \left\{ 1/(1 - |x|) \right\} + \theta(x) \right];$$

he showed that $0 \leq \theta(x) \leq 8/3$. We will prove the better result contained in the following

THEOREM 2. *If $\theta(x)$ is defined by (5), then*

$$1 \leq \theta(x) \leq \log_2(2eK/\log_2 e) < \log_2 4.90513 < 2.29430.$$

The lower bound is the best possible constant.

Finally we will prove the following

THEOREM 3. *If K^+ and $\theta^+(x)$ are defined analogously for the class Z^+ of even functions in Z , then $K^+ = 5/4$ (this result is known [1]) and $1 \leq \theta^+(x) \leq 5/3$.*

2. Proof of Theorem 1. We begin by stating an easily proved result of Brudnyi [1].

LEMMA 1. *If $f(x) \in Z$, if $-1 \leq a < b \leq +1$, if $\xi = (2x - a - b)/(b - a)$, and if the function $\phi(\xi)$ is defined on I so that*

$$(6) \quad f(x) = \frac{(x - a)f(b) + (b - x)f(a)}{b - a} + \frac{(b - a)\phi(\xi)}{2},$$

then $\phi(\xi) \in Z$, so that $|\phi(\xi)| \leq K$.

Theorem 1 will be a consequence of the following lemma, which describes a one-parameter family of uniform moduli of continuity for Z .

LEMMA 2. *Suppose that $0 < \delta < 2/3$ and that $f(x) \in Z$. Then*

$$(7) \quad |f(x) - f(y)| \leq \left\{ (K/2\delta) + \log_2(4\delta/|x - y|) \right\} |x - y|$$

for all x and y in I .

If $f(x)$ and $f(y)$ do not have opposite signs, it is obvious from (2) that

$$(8) \quad |f(x) - f(y)| \leq K.$$

Otherwise, $x \neq y$ (we may assume without loss of generality that $y < x$), and there exists a point z such that $y < z < x$ and $f(z) = 0$. If we use (6) when $(a, b) = (-1, z)$ and then when $(a, b) = (z, 1)$, it follows that $|f(y)| \leq (1 + z)K/2$, $|f(x)| \leq (1 - z)K/2$, so that (8) is true in this case also.

If $y \in I$ and $2x - y \in I$, it follows from (1) when $M = 1$ (all subsequent references to (1) will assume without comment that $M = 1$) and (8) that

$$(9) \quad 2|f(x) - f(y)| \leq |f(2x - y) - f(y)| + |2f(x) - f(2x - y) - f(y)| \\ \leq K + 2|x - y|.$$

If we interchange x and y , we see that (9) also holds when $x \in I$ and $2y - x \in I$. Because $2x - y = (x + y)/2 + 3(x - y)/2$ and $2y - x = (x + y)/2 - 3(x - y)/2$, it is clear that at least one of $2x - y$ and $2y - x$ must lie in I if $x \in I$, $y \in I$ and $|x - y| \leq 2/3$. In this case, therefore,

$$(10) \quad |f(x) - f(y)| \leq |x - y| + (K/2).$$

On the other hand, (10) is an immediate consequence of (8) when $(x - y) \geq 2/3$, if we recall that $K < 4/3$. We conclude that (10) is true for all x and y in I .

Now suppose that $0 < \delta < 2/3$, and that $|x - y| \geq \delta$. It follows from (10) that $|f(x) - f(y)| \leq \{1 + (K/2\delta)\}|x - y|$. This is the case $p = 1$ of the assertion that (11)

$$|f(x) - f(y)| \leq \{p + (K/2\delta)\}|x - y| \quad \text{if } x \in I, y \in I \text{ and } |x - y| \geq \delta/2^{p-1}.$$

Assume that (11) is valid for a particular p , and that $|x - y| \geq \delta/2^p$. If in fact $|x - y| \geq \delta/2^{p-1}$, then the validity of (11) with p replaced by $p + 1$ is obvious. When $\delta/2^p \leq |x - y| \leq \delta/2^{p-1} < 2/3$, then either $2x - y$ or $2y - x$ is in I , and either $|(2x - y) - y| = 2|x - y| \geq \delta/2^{p-1}$ or $|(2y - x) - x| = 2|x - y| \geq \delta/2^{p-1}$. We infer either from (9), or from (9) with x and y interchanged, and the inductive hypothesis that $|f(x) - f(y)| \leq \{p + (K/2\delta)\}|x - y| + |x - y|$, so that (11) is true when p is replaced by $p + 1$. Hence (11) is true for any integer p .

The truth of Lemma 2 when $x \neq y$ (Lemma 2 is obviously true when $x = y$) now follows from (11) if we choose the integer p so that $\delta/2^{p-1} < |x - y| \leq \delta/2^{p-2}$.

If $\delta = 1/4$, the inequality (7) yields a modulus of continuity similar to that of Timan in (3) but with the constant $8/3$ deleted. If $\delta = 1/2$, the resulting modulus of continuity $h\{K + \log_2(2/h)\}$ is still better, because $K + 1 < 2K$. The best result of the form (7) is stated in Theorem 1, and occurs when $\delta = K/(2\log_2 e)$. We observe that this value of δ is such that $0 < \delta < 2/3$; in fact, $0.450545 < K/(2\log_2 e) < 0.451125$.

Proof of Theorems 2 and 3. If we use Theorem 1 when $x \neq 0$, we see that

$$|f(x)| = |f(x) - f(\operatorname{sgn} x)| \leq (1 - |x|) \log_2 [2eK / \{(1 - |x|) \log_2 e\}].$$

This result is also true when $x = 0$ because $\sup_{f \in Z} |f(0)| = 1$ [5]. It follows that the function $\theta(x)$, defined by (5), satisfies the right-hand inequality in Theorem 2. This is better than Timan's result that $\theta \leq 8/3$. (If the conjecture that $K = 13/10$ is valid [6], then $\theta(x) \leq \log_2 4.89885 < 2.29245$.) Moreover, we will show in the next section that $(1 - x) \log_2 \{2/(1 - x)\}$, hence also $(1 + x) \log_2 \{2/(1 + x)\}$, is in Z . It follows that $\theta(x) \geq 1$; Timan had proved only that $\theta(x) \geq 0$. Because $\theta(0) = 1$, the present result is best possible. This completes the proof of Theorem 2.

Somewhat better results can be obtained if we restrict attention to the class Z^+ of even functions in Z . If

$$K^+ = \sup_{f \in Z^+} \max_{x \in I} |f(x)|,$$

it is known that $K^+ = 5/4$ [1]. The argument in §2 then shows that $\omega^+(h) = h \log_2 \{2eK^+ / (h \log_2 e)\}$ is a uniform modulus of continuity on Z^+ . The constant $2eK^+ / \log_2 e$ is less than 4.710423, and its logarithm to the base 2 is less than 2.235854.

If $\theta^+(x)$ is defined for Z^+ as $\theta(x)$ is defined for Z , it follows by repeating the argument in the first paragraph of this section that $\theta^+(x) \leq 2.23586$. In order to show, as asserted in Theorem 3, that $\theta^+(x) \leq 5/3$, some additional analysis is required. We begin by proving the following lemmas.

LEMMA 3. *If the function $g_1(x)$ is defined so that $g_1(x) = 1 + x$ if $0 \leq x \leq 1/4$, $g_1(x) = 5/4$ if $1/4 \leq x \leq 3/8$, $g_1(x) = 2(1 - x)$ if $3/8 \leq x \leq 1/2$, $g_1(x) = 1$ if $1/2 \leq x \leq 1$, then $|f(x)| \leq g_1(x)$ if $f(x) \in Z^+$ and $0 \leq x \leq 1$.*

If $f(x) \in Z^+$, it follows from (1), when $\xi = -x$, $\eta = x$ and the earlier observation that $|f(0)| \leq 1$ for all f in Z , that

$$(12) \quad |f(x)| \leq 1 + |x|.$$

We infer from (1), when $\xi = 0$ and $\eta = 1$, that $|f(1/2)| \leq 1$ for all f in Z . If we use (1) when $\xi = 1/2$, $\eta = 2x - \xi$, and assume that $1/4 \leq x \leq 1/2$ (so that $0 \leq \eta \leq \xi$), it follows from (12) with x replaced by η that $2|f(x)| \leq 1 + |f(\eta)| + 1 - 2x \leq 5/2$, so that

$$(13) \quad |f(x)| \leq 5/4 \quad \text{if } 1/4 \leq x \leq 1/2.$$

Finally, we use (1) when $\xi = 1$, $\eta = 2x - 1$ and assume that $0 \leq x \leq 1$. It follows from (12) with x replaced by η that $2|f(x)| \leq |f(\eta)| + 2(1 - x) \leq 3 - 2x + |2x - 1|$, so that

$$(14) \quad |f(x)| \leq 2(1 - x) \quad \text{if } 0 \leq x \leq 1/2,$$

$$(15) \quad |f(x)| \leq 1 \quad \text{if } 1/2 \leq x \leq 1.$$

Lemma 3 is now an obvious consequence of the inequalities (12), (13), (14) and (15).

LEMMA 4. *Starting with the function $g_1(x)$ defined in Lemma 3, we define the sequence of functions $g_n(x)$ ($n = 1, 2, 3, \dots$) inductively so that $g_{n+1}(x) = g_n(x)$ if $0 \leq x \leq 1 - 2^{-n}$, $g_{n+1}(x) = 1 - x + 2^{-1}g_n(2x - 1)$ if $1 - 2^{-n} < x \leq 1$. Then $|f(x)| \leq g_n(x)$ if $f(x) \in Z^+$ and $0 \leq x \leq 1$.*

We have already proved in Lemma 3 that $|f(x)| \leq g_1(x)$ when $f(x) \in Z^+$ and $0 \leq x \leq 1$. If $|f(x)| \leq g_n(x)$ for a particular positive integer n , then $|f(x)| \leq g_{n+1}(x)$ obviously, when $0 \leq x \leq 1 - 2^{-n}$. If $x > 1 - 2^{-n}$, then $x \neq 0$ and it follows from (1) when $\xi = 1$, $\eta = 2x - 1$ and the inductive hypothesis that

$$\begin{aligned} 2|f(x)| &\leq 2(1 - x) + |f(2x - 1)| \\ &\leq 2(1 - x) + g_n(2x - 1) = 2g_{n+1}(x). \end{aligned}$$

This completes the induction and the proof of Lemma 4.

LEMMA 5. *Let Δ_n , Δ'_n , Δ''_n , and Δ'''_n be, respectively, the intervals $(1 - 2^{1-n}, 1 - 3 \cdot 2^{-1-n})$, $(1 - 3 \cdot 2^{-1-n}, 1 - 5 \cdot 2^{-2-n})$, $(1 - 5 \cdot 2^{-2-n}, 1 - 2^{-n})$ and $(1 - 2^{-n}, 1)$. Then*

$$\begin{aligned} g_n(x) &= 2^{2-n} + (n - 2)(1 - x) \quad \text{if } x \in \Delta_n, \\ g_n(x) &= 5 \cdot 2^{-1-n} + (n - 1)(1 - x) \quad \text{if } x \in \Delta'_n, \\ g_n(x) &= (n + 1)(1 - x) \quad \text{if } x \in \Delta''_n, \\ g_n(x) &= 2^{1-n} + (n - 1)(1 - x) \quad \text{if } x \in \Delta'''_n. \end{aligned}$$

When $n = 1$, Lemma 5 is an obvious consequence of the definition of $g_1(x)$. If Lemma 5 is true for a particular n , and x is in Δ_{n+1} , Δ'_{n+1} , Δ''_{n+1} , or Δ'''_{n+1} , then

$2x - 1$ is in Δ_n , Δ'_n , Δ''_n , or Δ'''_n . The definition, $g_{n+1}(x) = 1 - x + 2^{-1}g_n(2x - 1)$ and the inductive hypothesis then imply immediately that Lemma 5 is true for $n + 1$.

To see that $\theta^+(x) \leq 5/3$, it is now sufficient to prove that

$$|f(x)| \leq (1 - x)[(5/3) + \log_2\{1/(1 - x)\}]$$

when $0 \leq x \leq 1$. This is obvious when $x = 1$. If $0 \leq x < 1$, choose n so that $2^{-n} < 1 - x \leq 2^{1-n}$. The desired inequality follows at once from Lemmas 4 and 5 if also $x \in \Delta_n$. Even stronger results are true if $x \in \Delta'_n$ or $x \in \Delta''_n$; the constant $5/3$ is replaced by $\log_2 3$ or $\log_2(5/2)$, respectively.

The fact that $\theta^+(x) \geq 1$ follows from the assertion that $(1 - |x|)\log_2\{2/(1 - |x|)\} \in Z^+$ (to be proved in the next section). The proof of Theorem 3 will then be complete.

We now define the function $f^*(x)$ so that $f^*(x) = g_n(x)$ when $2^{-n} < 1 - x < 2^{1-n}$, $f^*(1) = 0$, and $f^*(x) = f^*(-x)$ when $-1 \leq x < 0$.

THEOREM 4. *If $f(x) \in Z^+$ and $x \in I$, then $|f(x)| \leq f^*(x)$. This result is the best possible because $f^*(x) \in Z^+$.*

The first sentence of the theorem is an obvious consequence of the definition of $f^*(x)$ and Lemma 4 when $-1 < x < +1$. It is also true when $x = \pm 1$ because both f and f^* vanish in this case. The fact that $f^* \in Z^+$ will be an immediate consequence of the following lemmas and the observation that $f^*(1) = 0$, $f^*(x) = f^*(-x)$.

LEMMA 6. *The function $g_n(x)$ is continuous when $0 \leq x \leq 1$.*

LEMMA 7. *The sequence $g_n(x)$ converges uniformly to $f^*(x)$ when $0 \leq x \leq 1$. Hence $f^*(x)$ is continuous on I .*

LEMMA 8. *The function $f^*(x)$ satisfies the inequality (1).*

Lemma 6 is implied by the formulas in Lemma 5 and the observation that $g_n(x) = g_{n-1}(x)$ if $0 \leq x \leq 1 - 2^{1-n}$. The same observation shows that $f^*(x) = g_n(x)$ if $0 \leq x < 1 - 2^{-n}$. Therefore, $f^*(x) = \lim g_n(x)$ if $0 \leq x < 1$. On the other hand,

$$\lim g_n(1) = \lim(2^{1-n}) = 0 = f^*(1),$$

so that $f^*(x) = \lim g_n(x)$ if $0 \leq x \leq 1$. To see that the convergence of $g_n(x)$ to $f^*(x)$ is uniform, as asserted in Lemma 7, we first infer from Lemma 5 that $g_n(x)$ is a monotone decreasing function of x when $x \geq 1/4$ and that $g_n(1 - 2^{-n}) = (n + 1)2^{-n}$. If $n > m$, it then follows when $x \geq 1 - 2^{-m}$ that $g_m(x) \leq g_m(1 - 2^{-m})$, and $g_n(x) \leq g_n(1 - 2^{-m}) = g_m(1 - 2^{-m})$, so that

$$|g_n(x) - g_m(x)| \leq 2g_m(1 - 2^{-m}) \leq (m + 1)2^{1-m}.$$

This inequality also holds when $0 \leq x < 1 - 2^{-m}$ because its left-hand side is zero. The uniform convergence of the sequence $g_n(x)$ is now evident.

Lemma 8 can be proved by a direct calculation of the left-hand side of the inequality (1) when ξ or $-\xi$ lies in one of the intervals Δ_n , Δ'_n , or Δ''_n , and η lies in one of the intervals Δ_m , Δ'_m , or Δ''_m . It is not necessary to consider separately the case when ξ and η are both negative because f^* is even. Similarly, it is not necessary to

consider separately the case when ξ and $(\xi + \eta)/2$ are both negative, although the necessity to locate the point $(\xi + \eta)/2$ in an interval Δ_r , Δ'_r , or Δ''_r , does give rise to a number of separate calculations for each n and m .

4. Proof that certain functions are in Z or Z^+ . Suppose that

$$f(x) = (1 - |x|) \ln\{2/(1 - |x|)\}$$

if $|x| < 1$, $f(-1) = f(+1) = 0$, that $\Delta(x, y) = f(x) - 2f\{(x + y)/2\} + f(y)$, that $R = [(x, y) | x \in I, y \in I, x \neq y]$, that $H(x, y) = \Delta(x, y)/(x - y)$ on R , and $\lambda = \sup_{(x, y) \in R} |H(x, y)|$. We will show that $\lambda = \ln 2$. It is then obvious that $f(x)/\ln 2 = (1 - |x|) \log_2\{2/(1 - |x|)\} \in Z^+$, as claimed at the end of the previous section.

In calculating λ we can assume without loss of generality that $x > y$ (otherwise, interchange x and y) and that $x \geq -y$ (otherwise, replace x and y by $-y$ and $-x$, respectively). Thus we can restrict attention to the subset of R that is the union of the sets R_1 and R_2 , in which $R_1 = [(x, y) | 0 \leq y < x \leq 1]$, $R_2 = [(x, y) | -1 \leq -x < y < 0]$.

On R_1 , it is easy to see that $\Delta(x, y) = (1 - x) \ln(1 + u) + (1 - y) \ln(1 - v)$, in which $u = (x - y)/2(1 - x)$, $v = (x - y)/2(1 - y)$. Then $0 < u$, $0 < v \leq 1/2$, so that $0 < \ln(1 + u) < u$, $-2v \ln 2 \leq \ln(1 - v) < -v$. Therefore, $-(x - y) \ln 2 \leq \Delta(x, y) < 0$, and

$$(16) \quad -\ln 2 \leq H(x, y) < 0 \quad \text{on } R_1.$$

(This same argument will establish (16) for the function

$$g(x) = 2(1 - x) \ln\{2/(1 - x)\}$$

and the region $R_3 \equiv [(x, y) | -1 \leq y < x \leq 1]$. Therefore,

$$g(x)/\ln 2 = 2(1 - x) \log_2\{2/(1 - x)\}$$

is in Z , as claimed in §3.)

On R_2 we see that

$$\Delta(x, y) = (1 - x) \ln \frac{2}{1 - x} - (2 - x - y) \ln \frac{4}{2 - x - y} + (1 + y) \ln \frac{2}{1 + y}.$$

If $H(x, y)$ has an extreme value at some interior point (x_0, y_0) of R_2 , then $\partial H/\partial x = \partial H/\partial y = 0$ at (x_0, y_0) , so that $(x - y)\partial\Delta/\partial x - \Delta = 0$ and $\Delta + (x - y)\partial\Delta/\partial y = 0$ at (x_0, y_0) . It follows that, if $(x, y) = (x_0, y_0)$,

$$(17) \quad \ln w - \{(1 + y) \ln(1 + y) - 2y \ln 2\}/(1 - y) = 0,$$

$$\ln w - \{(1 + x) \ln(1 + y) - 2x \ln 2 + 2(x - y)\}/(1 - x) = 0,$$

in which $w = (2 - x - y)^2/4(1 - x)$. If we eliminate $\ln w$ by subtraction and remove a common factor $2(x - y)/(1 - x)(1 - y)$, we see that

$$(18) \quad 1 - y_0 = \ln\{2/(1 + y_0)\}.$$

It is easy to see that (18) has a unique solution y_0 on the open interval $(-1, 0)$. When y is this value y_0 , it follows from (17) that

$$w = \exp\left[\{(1 + y) \ln(1 + y) - 2y \ln 2\}/(1 - y)\right] = 2 \exp(-1 - y),$$

so that x_0 is a solution to the quadratic equation

$$x_0^2 + 2x_0(4e^{-1-y_0} - 2 + y_0) + (2 - y_0)^2 - 8e^{-1-y_0} = 0.$$

The discriminant of this quadratic equation is $32e^{-1-y_0}h(y_0)$, in which $h(y) = 2e^{-1-y} - 1 + y$ is negative when $-1 < y < 0$ because $h(0) = 2e^{-1} - 1 < 0$, $h(-1) = 0$ and $h''(y) = 2e^{-1-y} > 0$. We conclude that $H(x, y)$ has no extreme value at any interior point (x_0, y_0) of R_2 .

The border of R_2 is the union of the subsets R'_2 and R''_2 , defined so that $R'_2 = [(x, y)|y = -x, 0 < x \leq 1]$, $R''_2 = [(x, y)|x = 1, 0 \leq y \leq 1]$. On R'_2 we see that

$$\begin{aligned} H(x, y) &= H(x, -x) = \frac{1-x}{x} \ln \frac{1}{1-x} - \ln 2 \\ &= 1 - \ln 2 - \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}. \end{aligned}$$

It follows that, if $0 < x \leq 1$,

$$-\ln 2 = H(1, -1) \leq H(x, -x) < \lim_{x \rightarrow 0+} H(x, -x) = 1 - \ln 2 < \ln 2.$$

On R''_2 we see that

$$H(x, y) = H(1, y) = -\ln \frac{4}{1-y} + \frac{1+y}{1-y} \ln \frac{2}{1+y},$$

so that $H(1, -1) = H(1, 0) = -\ln 2$. The function $H(x, y)$ therefore has an extreme value when $y = y_0$ is such that $dH(1, y)/dy = 0$. This equation is the same as (18). Consequently, $H(1, y_0) = -\ln\{4/(1-y_0)\} + 1 + y_0$. This is an increasing function of y_0 when $-1 < y_0 < 0$; hence $-\ln 2 < H(1, y_0) < 1 - \ln 4 < 0$. It follows that (16) holds when $(x, y) \in R''_2$.

If we combine our results for R_1 , R'_2 and R''_2 we conclude that $-\ln 2 \leq H(x, y) < 1 - \ln 2$ on R . Therefore, $|\Delta(x, y)| \leq |x - y| \ln 2$ on R , and hence also for all x and y in I . This completes the proof that

$$(1 - |x|) \log_2 \{2/(1 - |x|)\}$$

is in Z^+ .

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EG & G IDAHO, INC., P. O. BOX 1625, IDAHO FALLS, IDAHO 83415

Current address: Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439