UNIQUENESS OF REPRESENTING MEASURES

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ABSTRACT. We improve a little an old theorem of Gleason and Whitney.

Let Δ be the maximal ideal space of the ring $H^\infty(\mathbf{D})$ of bounded holomorphic functions in the open unit disc $\mathbf{D}$. In this brief note we point out that if μ and τ are positive measures in Δ that represent the same point in $\mathbf{D}$, and if neither has any mass in $\mathbf{D}$, then $\mu = \tau$. A long time ago Gleason and Whitney proved that $\mu = \tau$ if neither has any mass off the Shilov boundary of $H^\infty(\mathbf{D})$. The point here is that $\mathbf{D}$ is smaller than the complement of the Shilov boundary.

LEMMA. Let $\xi \in \mathbf{D}$ and let μ be a positive Borel measure in Δ with

(i) $\mu(\{|\hat{z}| < 1\}) = 0$;

(ii) $f(\xi) = \int \hat{f} d\mu$ for every f in $H^\infty(\mathbf{D})$. ($\hat{f}$ is the Gelfand transform of f .)

Then

(iii) $\mu(\{|\hat{f}| < 1\}) = 0$ if f is inner.

PROOF. Let $f \in H^\infty(\mathbf{D})$; then

$$(1) \quad f(z) = \sum_{j=0}^{\infty} \alpha_j \left(\frac{\xi - z}{1 - \bar{\xi}z} \right)^j, \quad |z| < 1.$$

Put $\varphi = (\xi - z)/(1 - \bar{\xi}z)$. Then by (1),

$$f = \sum_{j=0}^k \alpha_j \varphi^j + \varphi^{k+1} f_{k+1},$$

where $f_{k+1} \in H^\infty(\mathbf{D})$. This gives

$$(2) \quad \int |\hat{f}|^2 d\mu = \sum_{j=0}^k |\alpha_j|^2 + \int |\hat{f}_{k+1}|^2 d\mu.$$

Here we use (i) and (ii) plus $\varphi(\xi) = 0$. By (2),

$$(3) \quad \sum_{j=0}^{\infty} |\alpha_j|^2 \leq \int |\hat{f}|^2 d\mu.$$

If f is inner, the left side of this inequality is 1. On the other hand, $|\hat{f}| \leq 1$ (if f is inner). This proves (iii). $\square$

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COROLLARY. Let μ and τ be positive regular Borel measures in Δ that represent the same point in $\mathbf{D}$. If neither has any mass in $\mathbf{D}$, i.e. if both $\mu(\{|\hat{z}| < 1\})$ and $\tau(\{|\hat{z}| < 1\})$ vanish, then $\mu = \tau$.

PROOF. Put $\Omega = \bigcup_{f \text{ inner}} \{|\hat{f}| < 1\}$. If F is a compact set in Ω , then, by the Lemma, $\mu(F) = 0$; hence, by regularity, $\mu(\Omega) = 0$. Likewise τ has no mass in Ω . But by Newman's theorem [1, 3, or 4]

$$\Omega' = \text{the Shilov boundary of } H^\infty(\mathbf{D});$$

hence, by the theorem of Gleason and Whitney [1, 2, or 3], $\mu = \tau$. $\square$

By the Douglas-Rudin theorem the inner functions in $\mathbf{D}$ separate points in the Shilov boundary of $H^\infty(\mathbf{D})$. Using this and the fact that equality holds in (3) if f is inner, one may give an alternative proof of the Corollary.

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