A NOTE ON CARLESON MEASURES IN PRODUCT SPACES

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ABSTRACT. In this article we give a very simple proof of a basic theorem about the class BMO in the multiple parameter setting.

The purpose of this note is to give a very simple proof of the following result.

THEOREM. *Suppose $f \in L^\infty(\mathbf{R}^2)$ and let u denote its biharmonic extension to $\mathbf{R}_+^2 \times \mathbf{R}_+^2$. Then $|\nabla_1 \nabla_2 u|^2 y_1 y_2 dz_1 dz_2$ is a Carleson measure on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$.*

For the definition of Carleson measure, as well as notation which we use here, see Chang [1] and Chang-Fefferman [2].

The theorem above is due to Alice Chang [1]. Somewhat later alternative proofs were found by R. Fefferman [3] and E. M. Stein [4]. Stein's proof is already quite short and is based on identities for harmonic functions and Green's theorem, while ours is real variable in nature. What follows is the result of discussions and work with Alice Chang, R. R. Coifman, and Y. Meyer, and the author wishes to thank them.

Our proof is based on the following simple lemma:

LEMMA. *Let $\psi_i(x)$ for $i = 1, 2$ be a function on $\mathbf{R}^1$ supported in $|x| < 1$ with $\|\psi_i\|_1 = 1$, $\psi_i \in L^2$, and $\int \psi_i = 0$. Let $\Omega \subseteq \mathbf{R}^2$ be an open set of finite measure, and define*

$$\Psi_{y_1, y_2}(x_1, x_2) = y_1^{-1} y_2^{-1} \psi_1(x_1/y_1) \psi_2(x_2/y_2).$$

Finally, set

$$\mathcal{L}_\Psi(f) = \left(\iint_{S(\Omega)} |f * \Psi_{y_1, y_2}|^2 \frac{dt dy}{y_1 y_2} \right)^{1/2}.$$

Then

$$\mathcal{L}_\Psi(f) \leq C \left(\prod_{i=1}^2 \log(\|\psi_i\|_2) \right)^{1/2} \cdot \|f\|_\infty |\Omega|^{1/2}.$$

PROOF OF THE LEMMA. When (t_1, t_2, y_1, y_2) is in the Carleson region $S(\Omega)$, then

$$f * \Psi_{y_1, y_2}(t) = f_0 * \Psi_{y_1, y_2}(t), \quad \text{where } f_0 = f \chi_\Omega.$$

This gives

$$\mathcal{L}_\Psi(f) \leq \left(\iint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} |f_0 * \Psi_{y_1, y_2}(t_1, t_2)|^2 \frac{dt dy}{y_1 y_2} \right)^{1/2}.$$

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The usual argument with the Plancherel theorem shows this last expression to be

$$\begin{aligned} &\leq \|f_0\|_2 \left(\prod_{i=1}^2 \int_{\mathbf{R}^1} |\hat{\psi}_i(\xi)|^2 \frac{d\xi}{|\xi|} \right)^{1/2} \\ &\leq \|f\|_\infty |\Omega|^{1/2} \left(\prod_{i=1}^2 \int_{\mathbf{R}^1} |\hat{\psi}_i(\xi)|^2 \frac{d\xi}{|\xi|} \right)^{1/2}. \end{aligned}$$

But for $|\xi| < 1$, $|\hat{\psi}_i(\xi)| \leq |\xi|$ since $\hat{\psi}_i(0) = 0$ and $|\hat{\psi}'_i(\xi)| = |[x\psi_i(x)]^\wedge(\xi)| \leq 1$. This gives

$$\int_{|\xi| \leq 1} |\hat{\psi}_i(\xi)|^2 \frac{d\xi}{|\xi|} \leq 1.$$

Obviously, $|\hat{\psi}_i(\xi)| \leq 1$, so

$$\int_{1 < |\xi| < \|\psi_i\|_2^2} |\hat{\psi}_i(\xi)|^2 \frac{d\xi}{|\xi|} \leq c \log \|\psi_i\|_2.$$

Finally,

$$\int_{|\xi| > \|\psi_i\|_2^2} |\hat{\psi}_i(\xi)|^2 \frac{d\xi}{|\xi|} \leq \frac{1}{\|\psi_i\|_2^2} \int_{\mathbf{R}^1} |\hat{\psi}_i(\xi)|^2 d\xi = 1.$$

This concludes the proof of the Lemma.

PROOF OF THE THEOREM. We prove that for open sets Ω ,

$$\iint_{S(\Omega)} \left| \frac{\partial^2 u}{\partial y_1 \partial y_2} \right|^2 y_1 y_2 dz_1 dz_2 \leq C|\Omega|.$$

(The part of $\nabla_1 \nabla_2$ coming from $\partial^2 u / \partial y_1 \partial y_2$, etc., is handled similarly.) This amounts to estimating

$$\left(\iint_{S(\Omega)} |f * \Psi_{y_1, y_2}(t_1, t_2)|^2 \frac{dt dy}{y_1 y_2} \right)^{1/2} = \mathcal{L}_\Psi(f)$$

for $\Psi(x_1, x_2) = \psi(x_1)\psi(x_2)$, where

$$\psi(x) = \frac{1-x^2}{1+x^2} \cdot \frac{1}{1+x^2}.$$

For $k \geq 1$ let

$$\psi_k(x) = \psi(x) [\chi_{2^{k-1} \leq |x| < 2^k}(x) - c_k \chi_{|x| \leq 1}(x)],$$

where c_k is chosen so that $\int \psi_k = 0$. Then $c_k = O(2^{-k})$ and $\sum \psi_k = \psi$. For integers $k, j \geq 1$, set $\Psi_{k,j}(x_1, x_2) = \psi_k(x_1)\psi_j(x_2)$. Let $\tilde{\Omega}_{k,j} = \{M_s(\chi_\Omega) > 1/2^{k+j}\}$. Then, applying the argument of the Lemma with $\Psi_{k,j}$ replacing Ψ and $f\chi_{\tilde{\Omega}_{k,j}}$ replacing $f\chi_\Omega$, we get

$$(\sim) \mathcal{L}_{\Psi_{k,j}}(f) \leq C[\log 2^k \log 2^j]^{1/2} \cdot 2^{-(k+j)} \|f\|_\infty |\tilde{\Omega}_{k,j}|^{1/2}.$$

By the strong maximal theorem $|\tilde{\Omega}_{k,j}| \leq C2^{k+j}(k+j)|\Omega|$, and Minkowski's inequality gives $\mathcal{L}_\Psi(f) \leq \sum_{j,k} \mathcal{L}_{\Psi_{k,j}}(f)$. Taking the estimate of $|\tilde{\Omega}_{k,j}|$ into account and summing $(\sim)$ on k, j completes the proof.

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