PSEUDO-PONTRJAGIN CLASSES

YASUO MATSUSHITA

ABSTRACT. For a pseudo-Riemannian manifold we can construct a pseudo-Pontrjagin class as represented by a certain $\text{ad}(\text{SO}_0(p, q))$ -invariant form on the manifold so that it coincides with the Pontrjagin class of the manifold.

Let M be an n -dimensional, orientable, C^∞ -differentiable, pseudo-Riemannian manifold with a metric $\langle, \rangle$ of signature

$$(\underbrace{+ \cdots +}_p + \underbrace{- \cdots -}_q), \quad p \geq q, \quad p + q = n.$$

By orientability we mean that the structure group of the tangent bundle ξ of M is the identity component $\text{SO}_0(p, q)$ of the pseudo-orthogonal group $\text{O}(p, q)$. As an analogue of the Pontrjagin class for a Riemannian manifold, we can define for M a pseudo-Pontrjagin class $\bar{\mathbf{p}}$ as an element of the de Rham cohomology ring $H^*(M; \mathbf{R})$ which is represented by a certain $\text{ad}(\text{SO}_0(p, q))$ -invariant curvature polynomial. We show that such a class $\bar{\mathbf{p}}$ coincides with the Pontrjagin class $\mathbf{p}$ constructed on the intrinsic Riemannian structure of M . Such a coincidence has been reported only for the case of the 4-manifolds of metric signature $(+ + - -)$ [2, Proposition 2]. This paper heavily depends on Chern's analysis [1] of the generalized Gauss-Bonnet formula for the Euler characteristics of pseudo-Riemannian manifolds.

At each point of M we can choose a local pseudo-orthonormal basis $\{e^i\}$ ($i = 1, \dots, n$) of 1-forms so that $\langle e^i, e^j \rangle = \varepsilon_i \delta_{ij}$ (δ_{ij} : Kronecker delta), where $\varepsilon_i = 1$ for $1 \leq i \leq p$ and $\varepsilon_i = -1$ for $p+1 \leq i \leq n$. The pseudo-Riemannian curvature tensor $R = (R^{ij}{}_{km})$ (of type (2.2)) defines a linear transformation on the bundle $\bigwedge^2$ of 2-forms over M :

$$R(e^i \wedge e^j) \equiv \Omega^{ij} = \frac{1}{2} \sum_{k,m} R^{ij}{}_{km} e^k \wedge e^m,$$

where Ω^{ij} are the components of the curvature 2-form with values in the Lie algebra $\mathfrak{so}(p, q)$.

The $n \times n$ matrix $\Omega = [\Omega^{ij}]$ is antisymmetric: $\Omega + {}^t\Omega = 0$. The curvature 2-form $\Omega' \equiv [\Omega_i{}^j]$ of type (1.1) is related to Ω as follows:

$$\Omega = I_{p,q} \Omega', \quad I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix},$$

where $I_k = \text{diag}[+1, \dots, +1]$ ($+1$: k times). Thus, Ω' satisfies $I_{p,q} \Omega' + {}^t\Omega' I_{p,q} = 0$. Under the actions of the group $\text{SO}_0(p, q)$, the curvature is transformed in the

Received by the editors May 29, 1984.

1980 *Mathematics Subject Classification*. Primary 53C50, 57R20.

Key words and phrases. Pseudo-Riemannian manifolds, pseudo-Pontrjagin classes, Pontrjagin classes.

©1985 American Mathematical Society
0002-9939/85 \$1.00 + \$.25 per page

following way:

$$(1) \quad \Omega' \mapsto \bar{\Omega}' = g^{-1}\Omega'g,$$

$$(2) \quad \Omega \mapsto \bar{\Omega} = {}^t g \Omega g,$$

where $g \in \mathrm{SO}_0(p, q)$.

Let $\mathfrak{M}_n(\mathbf{R})$ denote the algebra of all real $n \times n$ matrices. We now define a function $f: \mathbf{R} \times \mathfrak{M}_n(\mathbf{R}) \rightarrow \mathbf{R}$ by

$$f(t, X) = (-1)^q \det(tI_{p,q} - (2\pi)^{-1}X).$$

For this function we have

LEMMA. $f(t, \Omega)$ is invariant under the actions of $\mathrm{SO}_0(p, q)$.

PROOF. First we must note that every element $g \in \mathrm{SO}_0(p, q)$ satisfies ${}^t g I_{p,q} g = I_{p,q}$. From the transformation rule (2) for Ω , together with the above relation, we have

$$\begin{aligned} f(t, \bar{\Omega}) &= (-1)^q \det(tI_{p,q} - (2\pi)^{-1}\bar{\Omega}) \\ &= (-1)^q \det(tI_{p,q} - (2\pi)^{-1} {}^t g \Omega g) \\ &= (-1)^q \det(t {}^t g I_{p,q} g - (2\pi)^{-1} {}^t g \Omega g) \\ &= (-1)^q \det({}^t g (tI_{p,q} - (2\pi)^{-1}\Omega) g) = f(t, \Omega). \end{aligned}$$

This completes the proof. $\square$

It is to be noted that $f(t, \Omega)$ is an $\mathrm{ad}(\mathrm{SO}_0(p, q))$ -invariant form in the following sense. Put

$$f'(t, \Omega') \equiv f(t, I_{p,q}\Omega') = f(t, \Omega).$$

Since Ω' is transformed according to (1), we easily see the invariance as follows:

$$\begin{aligned} f'(t, \bar{\Omega}') &= f'(t, g^{-1}\Omega'g) = f(t, I_{p,q}g^{-1}\Omega'g) \\ &= f(t, {}^t g I_{p,q}\Omega'g) \quad [\text{i.e., } I_{p,q}g^{-1} = {}^t g I_{p,q}] \\ &= f(t, {}^t g \Omega g) = f(t, \Omega) = f'(t, \Omega'). \end{aligned}$$

Associated with $f(t, X)$, we can define $\mathrm{ad}(\mathrm{SO}_0(p, q))$ -invariant polynomial functions $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_n$ on $\mathfrak{so}(p, q)$ by

$$f(t, X) \equiv \sum_{i=0}^n \bar{g}_i(X) t^{n-i} \quad \text{for } X \in \mathfrak{so}(p, q).$$

For a $2k$ th function $\bar{g}_{2k}$ there exists a unique closed $4k$ -form $\bar{\beta}_k$ on M such that $\pi^*(\bar{\beta}_k) = \bar{g}_{2k}(\Omega)$, where $\pi: \xi \rightarrow M$ is the projection. We denote by $\bar{\mathfrak{p}}_k$ the class of the $4k$ th de Rham cohomology ring $H^{4k}(M; \mathbf{R})$ represented by $\bar{\beta}_k$ and call it the k th pseudo-Pontrjagin class of M . Put $\bar{\mathfrak{p}} = 1 + \bar{\mathfrak{p}}_1 + \bar{\mathfrak{p}}_2 + \dots$ and call $\bar{\mathfrak{p}}$ the total pseudo-Pontrjagin class of M .

We now state the main theorem.

MAIN THEOREM. *Let M be an n -dimensional orientable pseudo-Riemannian manifold of metric signature (p, q) , $p \geq q$, $p + q = n$. Then the pseudo-Pontrjagin class $\bar{\mathfrak{p}}$ of M coincides with the Pontrjagin class $\mathfrak{p}$ of M .*

PROOF. In [1] Chern observed that at each point $x \in M$, the tangent space $\pi^{-1}(x)$ splits into the p -dimensional subspace $\pi_+^{-1}(x)$ and the q -dimensional subspace $\pi_-^{-1}(x)$ with respect to the indefinite metric of signature (p, q) , together with a Riemannian metric on M . Since such a splitting is continuous and global, this implies that M admits a continuous field of tangent q -planes (see also [3, §40]). Thus, we can always choose a local pseudo-orthonormal basis $\{e^i\}$ of 1-forms on M so that $e^1, \dots, e^p$ lie on $\pi_+^{-1}(x)$ and $e^{p+1}, \dots, e^n$ lie on $\pi_-^{-1}(x)$. Relative to this basis the curvature 2-form Ω can be written as a diagonal block matrix

$$\Omega = \begin{bmatrix} \overbrace{\Omega_{(+)}^p} & \overbrace{0^q} \\ 0 & \overbrace{\Omega_{(-)}^q} \end{bmatrix} \begin{matrix} p \\ q \end{matrix}.$$

It is important to note that in this form Ω is compatible with the intrinsic Riemannian structure on M [1]. Therefore, inserting such an Ω into $f(1, \Omega)$, we have

$$\begin{aligned} f(1, \Omega) &= (-1)^q \det(I_{p,q} - (2\pi)^{-1}\Omega) \\ &= (-1)^q \det(I_p - (2\pi)^{-1}\Omega_{(+)} \wedge \det(-I_q - (2\pi)^{-1}\Omega_{(-)}) \\ &= (-1)^q \det(I_p - (2\pi)^{-1}\Omega_{(+)}) \wedge \det(-I_q - (2\pi)^{-1}\Omega_{(-)}) \\ &= (-1)^q \det(I_p - (2\pi)^{-1}\Omega_{(+)}) \wedge \det(-I_q + (2\pi)^{-1}\Omega_{(-)}), \end{aligned}$$

where the last equality holds since ${}^t\Omega_{(-)} = -\Omega_{(-)}$.

If q is odd then the second part of the last expression becomes

$$\det(-I_q + (2\pi)^{-1}\Omega_{(-)}) = -\det(I_q - (2\pi)^{-1}\Omega_{(-)}),$$

and, hence,

$$\begin{aligned} f(1, \Omega) &= (-1)^{q+1} \det(I_p - (2\pi)^{-1}\Omega_{(+)}) \wedge \det(I_q - (2\pi)^{-1}\Omega_{(-)}) \\ &= \det(I_{p+q} - (2\pi)^{-1}\Omega) = \det(I_n - (2\pi)^{-1}\Omega). \end{aligned}$$

If q is even

$$\det(-I_q + (2\pi)^{-1}\Omega_{(-)}) = \det(I_q - (2\pi)^{-1}\Omega_{(-)}),$$

and, therefore,

$$\begin{aligned} f(1, \Omega) &= (-1)^q \det(I_p - (2\pi)^{-1}\Omega_{(+)}) \wedge \det(I_q - (2\pi)^{-1}\Omega_{(-)}) \\ &= \det(I_{p+q} - (2\pi)^{-1}\Omega) = \det(I_n - (2\pi)^{-1}\Omega). \end{aligned}$$

Thus, irrespective of type of signature, the pseudo-Pontrjagin class $\bar{\mathfrak{p}}$ represented by a form $\bar{\beta}$, with $\pi^*(\bar{\beta}) = f(1, \Omega)$, coincides with the Pontrjagin class $\mathfrak{p}$ which is represented by β , with $\pi^*(\beta) = \det(I_n - (2\pi)^{-1}\Omega)$. $\square$

Although we treated the (pseudo-)Pontrjagin class of the tangent bundle of M with an indefinite metric, these results are easily applicable to general vector bundles with indefinite metrics.

I would like to thank Professor Chiaki Ihara for valuable discussions and encouragement.

REFERENCES

1. S. S. Chern, *Pseudo Riemannian geometry and the Gauss-Bonnet formula*, Acad. Brasil. Ciencias **35** (1963), 17-26.
2. Y. Matsushita, *Thorpe-Hitchin inequality for compact Einstein 4-manifolds of metric signature $(+ + - -)$ and the generalized Hirzebruch index formula*, J. Math. Phys. **24** (1983), 36-40.
3. N. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, N. J., 1951.

DEPARTMENT OF APPLIED MATHEMATICS AND PHYSICS, FACULTY OF ENGINEERING,
KYOTO UNIVERSITY, KYOTO, JAPAN