ESSENTIAL VERSUS #-SPECTRUM FOR SMOOTH DIFFEOMORPHISMS

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ABSTRACT. J. Robbin conjectures in his 1972 survey article (Bull. Amer. Math. Soc. **78**, 923–952) that the “essential” and #-spectra are identical for *all* C^1 -diffeomorphisms. If so, the stability conjecture of S. Smale follows. A #-spectrum may be attached to any orbit or invariant set and is a generalization of the set of eigenvalues of Tf at a fixed point. The essential spectrum is the closure of this spectrum, restricted to the periodic set of f . So Robbin’s conjecture meant that the periodic orbits carry the growth rate behavior of their closure. A counterexample is constructed and other conjectures made.

A counterexample to the 1972 conjecture of J. Robbin [Ro], that the essential and #-spectra are identical for *all* C^1 -diffeomorphisms, is constructed in this note.

The #-spectrum of a C^1 -diffeomorphism, $f: M^m \rightarrow M^m$, is the total spectrum of the operator on bounded C^0 -vector fields, $f^\#: C^0(M, TM) \rightarrow C^0(M, TM)$ given by

$$f^\#(\eta) = Tf^{-1} \circ \eta \circ f.$$

The #-spectrum may be naturally restricted to orbits of f or to any invariant set of f . In the terminology of Robbin’s survey paper [Ro], the “essential spectrum” of f is the closure of the #-spectrum restricted to the periodic set of f , $\text{per } f$.

In 1970 J. Franks showed that when f is structurally stable (see [Sm1 and PM] for definitions and background), the essential spectrum contains no complex number of modulus one [Fr]. It follows from the combined work of J. Kupka [Ku], S. Smale [Sm2], M. M. Peixoto [Pe], and C. Pugh [Pu1, Pu2], that structural stability implies “weak Axiom A”; that is, the #-spectrum over $\text{per } f$ contains no complex numbers of modulus one ($\text{per } f$ is hyperbolic), and $\overline{\text{per } f}$ equals the nonwandering set of f , $\Omega(f)$. So, were the essential spectrum known to equal the #-spectrum over $\overline{\text{per } f}$, then from the structural stability of f , it would follow that $\Omega(f)$ is hyperbolic and $\overline{\text{per } f} = \Omega(f)$. In the standard terminology of dynamical systems, from Robbin’s conjecture, S. Smale’s 1967 conjecture that structural stability implies Axiom A would follow [Sm1].

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The counterexample constructed here is not structurally stable, and so Robbin's proposed route to Smale's "Stability Conjecture" need only be amended. The example is C^∞ -smooth and lives on $\mathbf{R}^2$. Several questions and conjectures precede the Appendix, one of which provides an alternate route to the stability conjecture.

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The #-spectrum. When $\Lambda \subseteq M$, let $C_\Lambda = C(\Lambda, TM|_\Lambda)$ denote the space of all bounded continuous vector fields of M restricted to Λ . If $f(\Lambda) \subseteq \Lambda$, the C^1 -diffeomorphism $f: M \rightarrow M$ induces on C_Λ the #-map $f_\Lambda^\#: C_\Lambda \rightarrow C_\Lambda$ by $f^\#(\eta) = Tf^{-1} \circ \eta \circ f$. (Robbin calls the inverse of this map the *adjoint operator*.) Again, the total spectrum of $f_\Lambda^\#$, $\sigma(f_\Lambda^\#) \subseteq \mathbf{C}$, is the #-spectrum for $f|_\Lambda$. Occasionally, the notation $\sigma(f^\#|_\Lambda) = \sigma(f_\Lambda^\#)$ will be used. When $p \in M$, denote by $\sigma_p(f^\#) = \sigma(f^\#|_{o(p)})$, where $o(p) = \{f^n(p)\}_{n \in \mathbf{Z}}$ is the orbit of p . As is standard, $\text{per } f$ denotes the set of f -periodic points of M . The *essential spectrum* of $f^\#$ (or of f) is $\sigma_{\text{ess}}(f^\#) = \overline{\sigma(f^\#|_{\text{per } f})}$. J. Robbin's "essential spectrum conjecture" is that $\sigma_{\text{ess}}(f^\#) = \sigma(f^\#|_{\text{per } f})$.

The *resolvent set* of $p \in M$, $\text{Res}(p) \subseteq \mathbf{C}$, is the set of $\lambda \in \mathbf{C}$ such that $T_p M = E_{p,\lambda}^s \oplus E_{p,\lambda}^u$, where

$$E_{p,\lambda}^s = \left\{ v \in T_p M : |\lambda|^{-n} \|T_p f^n(v)\| \rightarrow 0 \text{ as } n \rightarrow \infty \right\},$$

$$E_{p,\lambda}^u = \left\{ v \in T_p M : |\lambda|^n \|T_p f^{-n}(v)\| \rightarrow 0 \text{ as } n \rightarrow \infty \right\}.$$

Here $|\cdot|$ is the usual modulus on $\mathbf{C}$ and $\|\cdot\|$ can be any Riemannian metric on TM . Occasionally, the notations $E_{p,\lambda}^s(T_p f)$ and $E_{p,\lambda}^u(T_p f)$ will also be used. When $\Sigma \subseteq \mathbf{C}$, the *annular hull* of Σ , $\text{Ann}(\Sigma)$, is the union of all circles centered at 0 passing through Σ .

The first lemma is a consequence of work by C. Chicone and R. Swanson [CS], R. Sacker and G. Sell [SS], R. Johnson [Jo], as well as J. Selgrade [Se]. Their combined work lays much of the ground work for the study of #-spectra of which this manuscript requires only a portion.

LEMMA 1. (a) $\sigma(f_\Lambda^\#) = \bigcup_{p \in \Lambda} \sigma_p(f^\#)$.

(b) $\text{Ann}(\sigma_p f^\#) = \mathbf{C} \setminus \text{Res}(p)$.

As an example, when p_0 is fixed by f , or $\Lambda = \{p_0\}$, then $C_\Lambda \approx T_{p_0} M$. Thus, $\sigma(f_\Lambda^\#) = \sigma(T_{p_0} f)$ and is finite. On the other hand, when $|\lambda|^{-n} \|T_p f^n(v)\| \rightarrow 0$ for some $\lambda \in \mathbf{C}$, then $|e^{i\theta} \lambda|^{-n} \|T_p f^n(v)\| \rightarrow 0$ as well. Likewise is the case when $|\lambda|^n \|T_p f^{-n}(v)\| \rightarrow 0$. Consequently, $\text{Res}(p_0)$ and hence $\mathbf{C} \setminus \text{Res}(p_0)$ is annular, and, in fact, the finite union of circles.

Now, let $\phi_t: M \rightarrow M$ be a C^1 -smooth flow generated by the vector field $X \in C_M$. When $p_0 \in M$ is periodic in ϕ_t with period $\tau > 0$, there exists a *transversal* π through p_0 , that is, a C^1 -imbedded submanifold such that for all $p \in \pi$, $T_p M = T_p \pi \oplus \chi_p$.

Here χ_p denotes the 1-dimensional subspace of $T_p M$ generated by $X(p)$. Furthermore, there exist open subsets, $p_0 \in U \subseteq V \subseteq \pi$ and a C^1 -first return map, $\rho: U \rightarrow V$, where $\rho(q)$ is the first forward reintersection of $q \in U$ with V . (J. Palis and W. deMello's new book is a good reference for background material [PM].)

LEMMA 2. If $p \in \text{per } f$ has period τ and $\rho: U \rightarrow V$ is as above, then

$$\sigma_p(\phi_\tau^\#) = \sigma(T_p \rho) \cup \{1\},$$

where $\sigma(T_p \rho)$ is the spectrum of the linear map $T_p \rho: T_p \pi \rightarrow T_p \pi$.

PROOF. $\text{proj}_\pi: T_p M \rightarrow T_p \pi$ is linear projection. Notice, $T_p \phi_\tau = (T_\pi \phi_\tau)_p + (T_\chi \phi_\tau)_p$, where $(T_\pi \phi_\tau)_p = \text{proj}_\pi \circ [(T_\pi \phi_\tau)|_{T_p \pi}]$ and $(T_\chi \phi_\tau)_p = (T_\chi \phi_\tau)|_{\chi_p}$. Again χ_p is generated by $X(p)$. The return map ρ equals ϕ_τ followed by projection along orbits of ϕ_τ . So $(T_\pi \phi_\tau)_p = T_p \rho$. In the second factor, $(T_\chi \phi_\tau)_p$ is the identity map on χ_p . Since ϕ_τ fixes p , $C_{o(p)} \approx T_p M$. Thus

$$\sigma_p(\phi_\tau^\#) = \sigma(T_\pi \phi_\tau)_p = \sigma(T_\pi \phi_\tau)_p \cup \sigma(T_\chi \phi_\tau)_p = \sigma(T_p \rho) \cup \{1\}.$$

The diffeomorphism ϕ_1 on $\mathbf{R}^2$. The counterexample of this note is the time-one map of a C^∞ -flow ϕ_t on $\mathbf{R}^2$. The Appendix contains a sketch of ϕ_t together with its detailed construction.

The lemma to follow summarizes the key properties of this flow, some of which for the sake of facilitating later discussions are quite specific.

LEMMA 3. There exists on $\mathbf{R}^2$ a C^∞ -flow ϕ_t generated by the vectorfield X such that:

(a) ϕ_t has two fixed points $p_0 = (1, 0)$ and $q_0 = (2, 0)$; p_0 is a saddle point while q_0 is a center. In a square region having p_0 at one corner, X is linear. One side of S_1 is contained within $W_{p_0}^u$ (the unstable manifold of p_0), another within $W_{p_0}^s$. The eigenvalues of $(D\phi_1)_{p_0}$ are $\{1/e, e\}$.

(b) $W_{p_0}^u \cap W_{p_0}^s$ is a topologically imbedded circle denoted W . This circle bounds the open disk D , which in turn contains the arc $(1, 2] \subset \{x\text{-axis}\}$. The angle at p_0 between this arc and $W_{p_0}^s$ is $\pi/4$. The nonwandering set $\Omega(\phi_1) = B = D \cup W$.

(c) All $p \in D$ are periodic in ϕ_t . Their periods are denoted $\tau(p)$. Furthermore, for all $p \in D$,

$$\int_0^{\tau(p)} \text{div } X(\phi_t(p)) dt = 0.$$

(d) On $W \setminus \{p_0\}$, $D\phi_1$ preserves a continuous splitting $(TH)|_W = TW \oplus N$. In a neighborhood about p_0 , $TW \perp N$.

The next lemma contains the spectral properties of $\phi_1^\#$ on B .

LEMMA 4. (a) For all $p \in D$, $\text{Ann } \sigma_p(\phi_1^\#) = \text{Ann}(1)$.

(b) $\sigma_{p_0}(\phi_1^\#) = \{1/e, e\}$.

(c) For all $p \in W \setminus \{p_0\}$, $\text{Ann } \sigma_p(\phi_1^\#) = \text{Ann}[1/e, e]$.

PROOF. At the center, $\sigma_{q_0}(\phi_1^\#) = \sigma(D\phi_1)_{q_0} \subseteq \text{Ann}(1)$. Let $p \in D \setminus \{q_0\}$. Since $X(p) \neq 0$ and p is periodic in ϕ_t , there exists a smooth transversal π to ϕ_t at p and a

first-return map ρ , near p . Assign to π the smooth coordinate system $z: \pi \rightarrow \mathbf{R}$ for which $z(p) = 0$. Then the classic Liouville Theorem [HS] is that

$$\left(\frac{d\rho}{dz}\right)_{z=0} = \exp\left[\frac{1}{\tau(p)} \int_0^{\tau(p)} \operatorname{div} X(\phi_t(p)) dt\right].$$

Applying Lemma 3(c), $(d\rho/dz)_{z=0} = 1$. Equivalently as a map from $T_p\pi \rightarrow T_p\pi$, $\sigma(T\rho)_p = \{1\}$. By Lemma 2, $\sigma_p(\phi_{\tau(p)}^\#) = \{1\}$. So when $\lambda \in \sigma_p(\phi_1^\#)$, $\lambda^{\tau(p)} = 1$ or $\lambda \in \operatorname{Ann}(1)$, the unit circle in $\mathbf{C}$.

Because p_0 is a fixed point of ϕ_1 , as in the proof of Lemma 2, $\sigma_{p_0}(\phi_1^\#) = \sigma(D\phi_1)_{p_0} = \{1/e, e\}$.

Lemma 3(d) is now used to establish (c): $(D\phi_1)$ preserves the continuous splitting $TH|W = TW \oplus N$. Thus for all $p \in W \setminus \{p_0\}$, $\sigma_p(\phi_1^\#) = \sigma_p(\phi_1^\#|C^0(W, TW)) \cup \sigma_p(\phi_1^\#|C(W, N))$. Let $v \in T_pW$. There exists $n_0 \geq 0$ such that for all n for which $|n| > n_0$, $\phi_n(p) \in S_1$ (the square next to p_0 on which X is linear). Then for all $n > n_0$, $\|(D\phi_n)_p(v)\| = ce^{-n}\|v\|$ for some constant c . Thus $|\lambda|^{-n}\|(D\phi_n)_p(v)\| = c|\lambda e|^{-n}\|v\| \rightarrow 0$ as $n \rightarrow \infty$, only when $|\lambda| > 1/e$. Similarly, $|\lambda|^n\|(D\phi_n)_p(v)\| = c|\lambda e^{-1}|^n\|v\| \rightarrow 0$ as $n \rightarrow \infty$, only when $|\lambda| < e$. So by Lemma 1, $\operatorname{Ann}[\sigma_p(\phi_1^\#|C^0(W, TW))] = \operatorname{Ann}(1/e, e)$.

Now consider $v \in N_p$. There exists $n_0 > 0$ such that for all n with the property $|n| > n_0$, $\phi_n(p) \in S_1$. Notice then for all $n > n_0$, $\|(D\phi_n)_p(v)\| = e^n\|v\|$ and $\|(D\phi_{-n})_p(v)\| = ce^{-n}\|v\|$. Thus $|\lambda|^{-n}\|(D\phi_n)_p(v)\| = c|\lambda^{-1}e|^n\|v\| \rightarrow 0$, as $n \rightarrow \infty$, only when $|\lambda| > e$ and $|\lambda|^n\|(D\phi_n)_p(v)\| = c|\lambda e|^n\|v\| \rightarrow 0$ only when $|\lambda| < 1/e$. So again by Lemma 1, $\operatorname{Ann}[\sigma_p(\phi_1^\#|C^0(W, N))] = \operatorname{Ann}[1/e, e]$. Because $D\phi_1$ preserves the splitting $(TH|W) = TW \oplus N$, $\operatorname{Ann}[\sigma_p(\phi_1^\#)] = \operatorname{Ann}[1/e, e]$ as claimed.

THEOREM. *There exists a C^∞ -diffeomorphism $\phi_1: \mathbf{R}^2 \rightarrow \mathbf{R}^2$, the time-one map of a C^∞ -flow, such that:*

- (a) $\operatorname{Ann} \sigma_{\text{ess}}(\phi_1^\#) = \operatorname{Ann}(1) \cup \{1/e, e\}$.
- (b) *There exists $q \in \overline{\operatorname{per} \phi_1}$ such that $\sigma_q(\phi_1^\#) = \operatorname{Ann}[1/e, e]$.*
- (c) $\sigma_{\text{ess}}(\phi_1^\#) \subseteq \sigma(\phi_1^\#|\overline{\operatorname{per} \phi_1})$.

PROOF. Let ϕ_1 be as constructed. (a) follows from Lemma 4(a) and 4(b). By Lemma 4(c) the point q may be any point on the saddle loop $W \setminus \{p_0\}$ while (c) follows from (a) and (b).

REMARKS AND CONJECTURES. (1) The nature of ϕ_1 and of $\sigma(\phi_1^\#)$ inspires the following conjecture: Let $f: M \rightarrow M$ be a C^1 -diffeomorphism of an m -manifold. Then for all $p \in \overline{\operatorname{per} f}$, $\sigma_p(f^\#) \cap \sigma_{\text{ess}}(f^\#) \neq \emptyset$.

(2) An amended route to Smale's conjecture is provided by the following conjecture: Again for $f: M \rightarrow M$, a C^1 -diffeomorphism, $\sigma(f^\#|\overline{\operatorname{per} f}) \cap \operatorname{Ann}(1) = \sigma(f^\#|\overline{\operatorname{per} f}) \cap \operatorname{Ann}(1)$. The example of this note does not counter this conjecture, nor do several shear-suspensions of ϕ_1 constructed by the author.

(3) One very important question remains: Is Robbin's conjecture true when structural stability is assumed.

Appendix. A flow ϕ_t , the existence of which was claimed in Lemma 2, is now constructed in more detail. (The author is aware that though quadratic vector fields,

such as $X = \begin{pmatrix} y \\ (x-1)(2-x) \end{pmatrix}$, generates flows ϕ_t with the spectral properties of Lemma 4, other requirements are not met.)

(I) The only fixed points of the C^∞ -flow ϕ_t on H are $p_0 = (1, 0)$ and $q_0 = (2, 0)$.

(II) Near p_0 , X is affine and has the form

$$X \begin{pmatrix} x \\ y \end{pmatrix} = R^{-1}AR \begin{pmatrix} x - 1 \\ y \end{pmatrix},$$

where

$$R = \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So p_0 is a saddle point of ϕ_1 . $(D\phi_1)_{p_0}$ has eigenvalues $\{1/e, e\}$.

(III) $W^u(p_0) \cap W^s(p_0)$ is the circle W . Such sets W are sometimes called “saddle loops”. W bounds the open disk D . Designate $B = D \cup W$.

(IV) B contains the square S_1 which is the rotation about p_0 by $-\pi/4$ of $[1, 3/2] \times [0, 1/2]$. On S_1 , $X = R^{-1}AR$. So p_0 is at one corner of S_1 , two sides of S_1 are contained within W , on one side ϕ_t ingresses, and on the last side ϕ_t egresses.

(V) A simple check concludes that the Poincaré map through S_1 is the identity map in W -normal coordinates; that is, if p lies on the S_1 -ingress set a distance s from W , then under ϕ_t , p will exit S_1 a distance s from W . Referring to Figure 1, “half” the S_1 -egress set is now connected to “half” the ingress set via a C^∞ -flow box S_2 . In doing so, the Poincaré map through S_2 is also the identity map in W -normal coordinates. Thus on $S_1 \cup S_2$, ϕ_t is assumed C^∞ and all orbits are periodic. (Remark. In general, the Poincaré map through a linear saddle square is of the form $s \rightarrow s^a$. In our case $a = 1$. When $a \neq 1$, this Poincaré map cannot be extended to a diffeomorphism at $W^s(p_0)$; hence no flow box S_2 exists which smoothly extends the flow on S_1 and closes up all orbits.)

FIGURE 1

(VI) In (x, y) -coordinates, $X(x, y) = (X_1(x, y), X_2(x, y))$. On S_1 , X has the following symmetry:

$$X_1(x, y) = -X_1(x, -y), \quad X_2(x, y) = X_2(x, -y).$$

S_2 is assumed symmetric about the x -axis, and the same X -symmetry is maintained. Thus on $S_1 \cup S_2$,

$$\frac{\partial X_1}{\partial x}(x, y) = -\frac{\partial X_1}{\partial x}(x, -y) \quad \text{and} \quad \frac{\partial X_2}{\partial y}(x, y) = -\frac{\partial X_2}{\partial y}(x, -y).$$

Consequently, $\operatorname{div} X(x, y) = -\operatorname{div} X(x, -y)$. But furthermore, this X -symmetry means that orbits of the X -flow ϕ_t are symmetric about the x -axis, and further that $|X(x, y)| = |X(x, -y)|$. Consequently, when $p \in S_1 \cap \{x\text{-axis}\}$, $\operatorname{div} X(\phi_t(p)) = -\operatorname{div} X(\phi_{\tau(p)-t}(p))$, where again $\tau(p)$ is the period of p . Let $u = (t - \tau(p))/2$. Then $\operatorname{div} X(\phi_u(p)) = -\operatorname{div} X(\phi_{-u}(p))$. Thus

$$\int_0^{\tau(p)} \operatorname{div} X(\phi_t(p)) dt = \int_{-\tau(p)/2}^0 \operatorname{div} X(\phi_u(p)) du + \int_0^{\tau(p)/2} \operatorname{div} X(\phi_u(p)) du = 0.$$

(VII) $W_{p_0}^u$ and $W_{p_0}^s$ are normal at p_0 , as is $\operatorname{bdry} S_1$ at W . In fact, inside S_1 , $D\phi_1$ preserves $(TW)^\perp$ along both $S_1 \cap W_{p_0}^u$ and $S_1 \cap W_{p_0}^s$. The vector field X is now smoothly reparametrized on S_2 , maintaining the X -symmetry in such a way that the transit times through S_2 are all equal to one. Then within S_2 , $D\phi_1$ preserves a splitting of $(TH|W) = TW \oplus N$ which at $\partial S_2 \cap W$ coincides with $TW + (TW)^\perp$.

(VIII) Let $\bar{S}$ be the union of all orbits of ϕ_t which pass through S_2 . Then $S_3 = D \setminus \bar{S}$ is a disk with boundary an orbit of ϕ_t . Smoothly extend ϕ_t on $\bar{S}$ to S_3 in such a way that all orbits of ϕ_t on S_3 are periodic, and q_0 is a center.

(IX) The nonwandering set $\Omega(\phi_1) = D \cup W$.

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