ACCUMULATION POINTS OF NOWHERE DENSE SETS IN H -CLOSED SPACES

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ABSTRACT. In this paper a ten year old problem by Kulpa and Szymański is settled by constructing an example of a minimal Hausdorff space without isolated points which has a point that is not the accumulation point of any nowhere dense subset of the minimal Hausdorff space. Also, a result by Kulpa and Szymański is extended by showing that a regular point in an H -closed space without isolated points is the accumulation point of some nowhere dense subset.

A decade ago, Kulpa and Szymański [KS] showed that each point in a compact Hausdorff space without isolated points is the accumulation point of a nowhere dense set and they asked if the result is true in the setting of minimal Hausdorff spaces without isolated points. They noted that the result is false for arbitrary H -closed spaces without isolated points. In this paper, we settle this question by giving an example of a minimal Hausdorff space without isolated points and a point which is not the accumulation point of any nowhere dense set. Also, we extend Kulpa and Szymański's result by showing that a regular point in an H -closed space without isolated points is the accumulation point of some nowhere dense subset.

First, a few preliminary results and definitions are needed. All spaces considered in this paper are assumed to be Hausdorff.

A space X is H -closed if X is closed in every space in which X is a subspace. An open set U in a space X is said to be *regular open* if $U = \text{int}(\text{cl } U)$. A space X is *semiregular* if $\{U \subseteq X: U \text{ is regular open}\}$ is a base for the topology of X and is *minimal Hausdorff* if there is no strictly coarser Hausdorff topology on the space.

Katětov [K] has shown that a space is minimal Hausdorff iff it is H -closed and semiregular. For a space X , let $\mu X = X \cup \{\mathcal{U}: \mathcal{U} \text{ is a free open ultrafilter on } X\}$. For each open set $U \subseteq X$, let $oU = U \cup \{\mathcal{U} \in \mu X \setminus X: \text{int}(\text{cl } V) \subseteq U \text{ for some } V \in \mathcal{U}\}$.

THEOREM 1. *Let X be a semiregular space. The following are true:*

(a) [B, F, S] *The family $\{oU: U \text{ is regular open in } X\}$ is a base for a minimal Hausdorff extension μX of X .*

(b) [PW] *For a regular open subset U of X , $\text{cl}_{\mu X} oU = oU \cup \text{cl}_X U$. $\square$*

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The space μX is called the *Banaschewski-Fomin-Sanin minimal Hausdorff extension* of a space X ; additional information about this extension can be found in [PV, PW₂].

For a space X , let $EX = \{\mathcal{U}: \mathcal{U} \text{ is a fixed open ultrafilter on } X\}$ and define $k: EX \rightarrow X$ by $k(\mathcal{U}) = \bigcap \{cl_X U: U \in \mathcal{U}\}$. For each open set $U \subseteq X$, let $OU = \{\mathcal{U} \in EX: U \in \mathcal{U}\}$. The family $\{OU: U \text{ open in } X\}$ forms a base for an extremally disconnected, Tychonoff topology on EX ; the pair (EX, k) is called the *absolute* of X (see [Wo]). Let $f: R \rightarrow T$ be a function between two spaces R and T . Recall that f is *perfect* if $f^{-1}(t)$ is compact for each $t \in T$ and f is closed; f is *irreducible* if for each closed set $A \subseteq R$, $A = R$ whenever $f[A] = T$; and f is θ -*continuous* if for each $r \in R$ and neighborhood U of $f(r)$, there is some neighborhood V of r such that $f[cl_R V] \subseteq cl_T U$.

THEOREM 2 [Wa]. *Let X be a space. The following are true:*

- (a) *The function $k: EX \rightarrow X$ is a perfect, irreducible, θ -continuous surjection.*
- (b) *The space X is H -closed iff EX is compact.* $\square$

Let X be a space and $p \in X$. Let $Y_p = EX/k^{-1}(p)$, i.e., let Y_p be the quotient space of EX with $k^{-1}(p)$ identified to a point, denoted as $\bar{p}$. Let $f: EX \rightarrow Y_p$ be the induced quotient map, i.e., $f(y) = y$ for $y \in EX \setminus k^{-1}(p)$ and $f(y) = \bar{p}$ for $y \in k^{-1}(p)$. Define $g: Y_p \rightarrow X$ by $g(y) = k(y)$ for $y \neq \bar{p}$ and $g(\bar{p}) = p$.

A point p in a space X is a *regular point* if for each neighborhood U of p , there is a neighborhood V of p such that $cl V \subseteq U$. The next result is easy to verify.

LEMMA 3. *Suppose X is H -closed and $p \in X$. The following are true:*

- (a) *The space Y_p is compact and zero dimensional.*
- (b) *The function $f: EX \rightarrow Y_p$ is perfect, irreducible, continuous and onto and $f(EX \setminus k^{-1}(p))$ is a homeomorphism onto $Y_p \setminus \{\bar{p}\}$.*
- (c) *The function $g: Y_p \rightarrow X$ is perfect, irreducible, θ -continuous and onto.*
- (d) *If p is a regular point of X , then g is continuous at $\bar{p}$.* $\square$

Lemma 3 is now used to extend the result by Kulpa and Szymański that each point in a compact space without isolated points is an accumulation point of some nowhere dense set.

THEOREM 4. *Let p be a regular point of an H -closed space X without isolated points. Then p is an accumulation point of some nowhere dense subset of X .*

PROOF. Let Y_p be defined as in Lemma 3. There is a nowhere dense set $A \subseteq Y_p$ such that $\bar{p}$ is an accumulation point of A . We can assume that $\bar{p} \notin A$. Using the fact that g is perfect and irreducible, it is straightforward to show that $g[A]$ is nowhere dense in X . Since $g^{-1}(p) = \{\bar{p}\}$, it follows that $p \notin g[A]$. Let V be a neighborhood of p . By the continuity of g at $\bar{p}$, there is a neighborhood W of $\bar{p}$ such that $g[W] \subseteq V$. As $W \cap A \neq \emptyset$, it follows that $V \cap g[A] \neq \emptyset$. So, p is an accumulation point of the nowhere dense set $g[A]$. $\square$

Let βX denote the Stone-Čech compactification of the Tychonoff space X . A point $p \in \beta X$ is called a *remote point* of X if $p \notin cl_{\beta X} A$ for each closed, nowhere dense subset A of X . By a result in [vD], βQ contains a remote point p of Q , where

$\mathbf{Q}$ is the space of rational numbers. The subspace $Y = \mathbf{Q} \cup \{p\}$ of $\beta\mathbf{Q}$ is Tychonoff and has the property that p is not the accumulation point of any nowhere dense subset of Y . So, μY is a reasonable prospect for a minimal Hausdorff space in which p is not an accumulation point of any nowhere dense set of Y . However, as Y is Lindelöf (and hence realcompact), Theorem 6 shows that the space μY is not a counterexample. Before proving Theorem 6, we need a lemma.

LEMMA 5. *Let Y be a realcompact space and let $q \in \beta Y \setminus Y$. Then there exists a zero-set Z of βY such that Z has no isolated points and $q \in Z \subseteq \beta Y \setminus Y$.*

PROOF. By 1.53 of [Wa], there exists a zero-set Z of βY such that $q \in Z \subseteq \beta Y \setminus Y$. Let $C = \beta Y \setminus Z$. Then $Y \subseteq C \subseteq \beta Y$, so $\beta C = \beta Y$ (see 1.49 of [Wa]), and hence $Z = \beta C \setminus C$. But since C is a cozero-set of βY , it is realcompact, and thus $\beta C \setminus C$ has no isolated points (see the argument in the proof of 3.3 of [P], for example). Hence, Z has no isolated points. $\square$

THEOREM 6. (a) *Let X be a semiregular space, and let p be a regular point of X . Then μX is regular at p .*

(b) *If X is a realcompact space without isolated points, then each point of μX is the accumulation point of a nowhere dense subset of μX .*

PROOF. To prove (a), let U be a regular open subset of X and let $p \in oU$. There is a regular open set in V in X such that $p \in V \subseteq \text{cl}_X V \subseteq U$. So $p \in oV \subseteq oU$. Since $\text{cl}_{\mu X} oV = (\text{cl}_X V) \cup oV$, it follows that $oV \subseteq \text{cl}_{\mu X} oV \subseteq oU$. So, μX is regular at p . To establish (b), note that as X is Tychonoff, it follows by Theorems 4 and 6(a) that each point of X is an accumulation point of some nowhere dense subset of μX . Let $r \in \mu X \setminus X$. It is known that EX is realcompact [Wo, Theorem 4.6] and that the function $f: EX \rightarrow X$ has a perfect irreducible θ -continuous extension $\bar{k}: \beta EX \rightarrow \mu X$ such that $\bar{k}|(\beta EX \setminus EX): \beta EX \setminus EX \rightarrow \mu X \setminus X$ is a homeomorphism (see [PV, PW₁, Wo]). So, $\bar{k}(q) = r$ for some $q \in \beta EX \setminus EX$. By Lemma 5, there exists a zero-set Z of βEX such that $q \in Z \subseteq \beta EX \setminus EX$ and Z has no isolated points. As $\bar{k}|(\beta EX \setminus EX)$ is a homeomorphism from $\beta EX \setminus EX$ onto $\mu X \setminus X$, it follows that $\bar{k}[Z]$ has no isolated points and $r \in \bar{k}[Z]$. As $\bar{k}[Z] \subseteq \mu X \setminus X$, it is nowhere dense in μX , and r is an accumulation point of $\bar{k}[Z]$. $\square$

To construct a minimal Hausdorff space without isolated points which has a point which is not the accumulation point of any nowhere dense subset, we will use a construction introduced by Herrlich in [H]. Let D be a dense subset of a space X and for $n \in \mathbf{N}$ ($\mathbf{N}$ is the set of positive integers), let $X(D^n) = (D \times \{0, 1, \dots, n-1\}) \cup (X \setminus D)$. If $A \subseteq X$, let $A^n = ((A \cap D) \times \{0, 1, \dots, n-1\}) \cup (A \setminus D)$. A set $U \subseteq X(D^n)$ is defined to be open in $X(D^n)$ if $U \cap (D \times \{i\})$ is open in $D \times \{i\}$ (with the product topology) for $i = 0, 1, \dots, n-1$, and $x \in U \cap (X \setminus D)$ implies $x \in V^n \subseteq U$ for some open set V in X .

LEMMA 7. *Let D be a dense subspace of a space X and $n \in \mathbf{N}$. The following are true.*

(a) *If X is H -closed, so is $X(D^n)$.*

(b) *If $n \geq 2$ and X is semiregular, then $X(D^n)$ is semiregular. In particular, if X is minimal Hausdorff, so is $X(D^2)$.*

PROOF. The proof is straightforward. $\square$

EXAMPLE 8. Let p be a remote point of $\beta\mathbf{Q}$. Then $D = \mathbf{Q} \cup \{p\}$ is dense in $\beta\mathbf{Q}$. Let $X = (\beta\mathbf{Q})(D^2)$. Clearly X has no isolated points and by Lemma 7, X is minimal Hausdorff. Let F be a closed, nowhere dense subset of X . Since $(\beta\mathbf{Q} \setminus D) \cup (D \times \{1\})$ is closed in X , then $(p, 0) \in \text{cl}_X(F \setminus \{(p, 0)\})$ iff $(p, 0) \in \text{cl}_X(F \cap (\mathbf{Q} \times \{0\}))$. Now, $F \cap (\mathbf{Q} \times \{0\}) = H \times \{0\}$ for some $H \subseteq \mathbf{Q}$, and H is closed and nowhere dense in $\mathbf{Q}$. Since $p \notin \text{cl}_{\beta\mathbf{Q}} H$, it follows that $(p, 0) \notin \text{cl}_X(F \cap (\mathbf{Q} \times \{0\}))$. So, X is a minimal Hausdorff space without isolated points and the point $(p, 0)$ is not the accumulation point of any nowhere dense subset of X . $\square$

COMMENT. Let A be a closed, nowhere dense subset of a metric space X without isolated points and let $p \in \mu X \setminus X$. By 1.2 in [PW₁], there is a regular open set U of X such that $A \subseteq \text{bd}_X U$; it follows that $\text{cl}_{\mu X} A = A$. Thus, each point $p \in \mu X \setminus X$ is a "remote point" of μX . If $D = X \cup \{p\}$, then the same proof as in the above example yields that $(\mu X)(D^2)$ is a minimal Hausdorff space and the point $(p, 0)$ is not the accumulation point of any nowhere dense subset of $(\mu X)(D^2)$. On the other hand, using Theorem 6(b) and the fact that metric spaces are realcompact (assuming the nonexistence of measurable cardinals), μX is not a counterexample.

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