WORDS PERIODIC OVER THE CENTER OF A DIVISION RING¹

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ABSTRACT. In generalization of a result of Herstein, the authors prove that, in a division ring with uncountable center, if any given nontrivial group word takes only values periodic over the center, then the division ring is commutative. Techniques include use of the result that a noncommutative division ring finite-dimensional over its center includes a nonabelian free group in its multiplicative group.

The first author has shown [7] that a noncommutative division ring with uncountable center contains a nonabelian free subsemigroup in its multiplicative group. In attempting to generalize his methods to prove the existence of a free subgroup, we instead derive a result limiting the existence of a group word, all of whose values over a division ring are periodic over the center. Specifically, we prove that no nontrivial example of such a word can exist for a noncommutative division ring with uncountable center. This generalizes a result of Herstein [2], which considers the specific case of the multiplicative commutator (as a group word). Our methods of proof include the use of a polynomial identity to reduce to the case of a finite-dimensional divisional ring and the use of a form of the Tits alternative in such a case.

The form we will use states that in a division ring finite-dimensional over its center, the multiplicative group either contains a nonabelian group or is commutative (i.e. we have a field). This is implicit in the work of Lichtman [5 and 6] (or see [1]). Or we may outline an argument as follows: consider the multiplicative group as a linear group over the center. Then by the original Tits alternative for fields [9], either the group contains a free subgroup or is solvable-by-locally finite (see Tits' Theorem 2(ii) for nonzero characteristic, or use Theorem 1 and substitute the intersection of the conjugates for zero characteristic). Now the normal solvable subgroup is central by Scott's result [8]. Then the division ring is locally finite over the center, and a result of Kaplansky [4] implies that the division ring is a field.

We now discuss our definitions. By the term *group word* we simply mean an element w of a free group, say on some set of generators. The *values* of a group word over a division ring are the results of substituting nonzero elements of the division ring for each of the generators appearing in the word. An element d of a division ring is said to be *periodic* over a subset S of the division ring if $d^n \in S$ for some integer $n > 0$.

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Before proceeding to the main result, we prove two lemmas. The first of these concerns the polynomial ring $T[\alpha]$ over a division ring T in a central indeterminate α ; this is a noetherian domain by the standard Hilbert Basis Theorem argument. The second applies to the case of nonzero characteristic and is analogous to Lemma 2 of Herstein's paper [2]. During these arguments we will at times use the notation $[x, y]$ for the *additive* commutator $xy - yx$.

LEMMA 1. *Let T be a division ring, $T[\alpha]$ the Ore ring of polynomials in a central variable α , and $T(\alpha)$ its quotient division ring of fractions. If $f \in T(\alpha)$ and if $f(z)$ is defined and central in T for an infinite number of z central in T , then f is central in $T(\alpha)$.*

PROOF. Certainly f is central if and only if each commutator $[f, y] = 0$ for $y \in T$. Thus it is sufficient to show that if $g \in T(\alpha)$ satisfies $g(z) = 0$ for infinitely many z central in T , then $g = 0$. But $g = g_1 g_2^{-1}$, where $g_1, g_2 \in T[\alpha]$, and so $g_1(z) = 0$ for infinitely many central z . The standard Vandermonde argument now shows that the coefficients of g_1 are all zero.

LEMMA 2. *Let D be a division ring with center Z and with characteristic $\kappa \neq 0$. Suppose there is a $a \in D$ such that $a^\kappa \in Z$ but $a \notin Z$. Then D^* contains a nonabelian free subgroup.*

PROOF. Define ad_x by $\text{ad}_x(y) = [x, y]$. Since the characteristic is κ , we get $(\text{ad}_a)^\kappa(x) = [a^\kappa, x]$ for any $x \in D$. Pick $x \in D$ and integer $\kappa > k \geq 1$ (since $a \notin Z$, $a^\kappa \in Z$) so that $\text{ad}_a^k(x) = y \neq 0$ but $\text{ad}_a^{k+1}(x) = 0$. Then putting $z = \text{ad}_a^{k-1}(x)$, we see that $[a, y^{-1}z] = 1$. Thus D contains a copy of the Weyl algebra, and so a copy of its classical division ring of quotients. In nonzero characteristic this subdivision ring of D is finite-dimensional over its center, and hence by our form of the Tits alternative contains a nonabelian free group in its multiplicative group. Thus D^* contains such a subgroup.

THEOREM. *Let D be a division ring with uncountable center Z and suppose that there is a nontrivial word w in a free group such that every value of w over D is periodic over Z . Then D is commutative.*

PROOF. Suppose that w involves k of the free generators of the free group. Let w' be the image of w under an embedding of the free group on k generators into the free group on 2 generators. Then for any two elements of D^* , the corresponding value of w' is a value of w corresponding to a substitution of a certain k elements of D^* into w . The result is still periodic over Z ; thus, replacing w by w' if necessary, we may assume $k = 2$.

Now let σ, τ be noncommuting variables over the prime field of D and let α, β be central variables. We may expand the substitution $w(1 + \alpha\sigma, 1 + \beta\tau)$ in power series in these variables, using geometric series. Since w is nontrivial, the result may be written

$$w(1 + \alpha\sigma, 1 + \beta\tau) = 1 + \alpha^p \beta^q w_1(\sigma, \tau) + R,$$

where w_1 is a nonzero polynomial over the prime field of D and where all terms in R have $(\alpha$ -degree, β -degree) greater than (p, q) in lexicographic ordering. That the substitution is not completely trivial may be seen by examining terms $\sigma^{i_1}\tau^{j_1} \cdots \sigma^{i_n}\tau^{j_n}$ with maximum n .

Let s, t be any elements of D . Allowing a, b to take values from Z , each of $w(1 + as, 1 + bt)$ must be defined and periodic over Z unless $a = -s^{-1}$ or $b = -t^{-1}$. For each $b \in Z, b \neq -t^{-1}$, there are uncountably many $a \neq -s^{-1}$, so there exists an $n = n(b) > 0$ such that $w(1 + as, 1 + bt)^n$ is central for uncountably many a . If we replace a by a central variable α , then Lemma 1 implies that $w(1 + \alpha s, 1 + bt)^n$ is central in $D(\alpha)$. Since this is true for every $b \neq -t^{-1}$, again there must be a fixed $N > 0$ so that $w(1 + \alpha s, 1 + bt)^N$ is central $D(\alpha)$ for uncountably many $b \neq -t^{-1}$. Finally we can apply Lemma 1 again, using a central variable β , to obtain that $w(1 + \alpha s, 1 + \beta t)^N$ is central in $D(\alpha)(\beta)$.

Now embed $D(\alpha)(\beta)$ in the Laurent series division ring $D((\alpha))((\beta))$. The image of $w(1 + \alpha s, 1 + \beta t)$ is $\bar{w} = 1 + \alpha^p \beta^q w_1(s, t) + \bar{R}$ as in our earlier notation, where all terms in $\bar{R}$ have higher degrees than (p, q) . Furthermore, $\bar{w}^N$ is still central. If the characteristic of D is zero, then the first two nonzero terms in $\bar{w}^N$ are $1 + N\alpha^p \beta^q w_1(s, t)$, so that $w_1(s, t)$ must be central. If the characteristic of D is $\kappa \neq 0$, then write $N = \lambda M$ where λ is a power of κ and M is prime to κ . Then

$$\bar{w}^N = \left[1 + (\alpha^p \beta^q w_1(s, t) + \bar{R})^\lambda \right]^M,$$

the first two terms of which are $1 + M\alpha^p \beta^q w_1(s, t)^\lambda$. Now we see that $w_1(s, t)^\lambda$ must be central. Since the assumptions of this theorem imply that D^* cannot contain a nonabelian free subgroup, we can apply Lemma 2 to conclude that $w_1(s, t)$ is central.

Thus $w_1(s, t)$ is central for every $s, t \in D$, giving a polynomial identity $[w_1(\sigma, \tau), \nu] = 0$ for D . We conclude that D is finite dimensional over Z [3, p. 226]. As before, D^* cannot contain a nonabelian free group. Then by the Tits alternative, we must have $D = Z$.

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