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GORENSTEIN ALGEBRAS AND THE CAYLEY-BACHARACH THEOREM

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ABSTRACT. This paper is an examination of the connection between the classical Cayley-Bacharach theorem for complete intersections in $\mathbf{P}^2$ and properties of graded Gorenstein algebras.

Introduction. It is known, if not well known, that the Cayley-Bacharach theorem for complete intersections in $\mathbf{P}^2$ is valid for 0-dimensional arithmetically Gorenstein subschemes of $\mathbf{P}^n$. More generally, we show that the result is valid for 0-dimensional subschemes of $\mathbf{P}^n$ having minimal Cohen-Macaulay type compatible with their Hilbert functions. The Cayley-Bacharach theorem in the Gorenstein case is a special instance of a theorem, interesting and technically useful in its own right, relating the Hilbert functions of linked subschemes of $\mathbf{P}^n$. Lastly we show that the 0-dimensional, arithmetically Gorenstein, reduced subschemes of $\mathbf{P}^n$ are characterized by the validity of the Cayley-Bacharach theorem and the symmetry of the Hilbert function.

Fixed notation. A denotes a *standard* $\mathbf{N}$*-graded* k*-algebra*, k a field: $A_0 = k$; $A = k[A_1]$; $\lambda(A_1) < \infty$. We use λ to denote k-linear dimension, reserving "dim" for dimension of rings or schemes, and δ to denote multiplicity for such algebras (or degree of the corresponding projective scheme). Note that $\delta(A) = \lambda(A)$ if $\dim A = 0$. We use I to denote a nonzero, nonunit, homogeneous ideal of A, and $J = \operatorname{ann} I$ ("ann" = annihilator). We assume always that A and A/I are CM (Cohen-Macaulay) and that $\operatorname{Ass}(A/I) \subset \operatorname{Ass}(A)$. Hence $\operatorname{Ass}(A/J) \subset \operatorname{Ass}(A)$. (Indeed, since the 0-ideal of A is unmixed of height 0, so is the annihilator of any nonzero ideal of A.) Therefore A/J is CM if $\dim A \leqslant 1$. In any case, A/J is CM if A is Gorenstein [**PS**, Proposition 1.3]. In our applications A/J will be CM for one of these two reasons.

Recall that, by definition, a ring R is Gorenstein provided that $R_{\mathfrak{p}}$ is a Gorenstein local ring for every prime ideal $\mathfrak{p}$ of R, and A is Gorenstein $\Leftrightarrow$ A_{A_1A} is a Gorenstein local ring [**AG**]. We refer to [**K**] for those properties of Gorenstein local rings which are used below without specific reference.

1. OBSERVATIONS. Assume that $A_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Ass}(A)$). Then:

(a) $\delta(A) = \delta(A/I) + \delta(A/J)$.

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(b) Suppose $\dim A = 0$, and let $N = \max\{t \in \mathbf{N} | A_t \neq 0\}$. Then

$$\lambda(A_t) = \lambda(I_t) + \lambda(J_{N-t}) = \lambda(A_{N-t}), \qquad 0 \leqslant t \leqslant N.$$

PROOF. (a) Since $A_{\mathfrak{p}}$ is a 0-dimensional Gorenstein local ring for $\mathfrak{p} \in \operatorname{Ass}(A)$,

$$\operatorname{length}(A_{\mathfrak{p}}) = \operatorname{length}(A_{\mathfrak{p}}/IA_{\mathfrak{p}}) + \operatorname{length}(A_{\mathfrak{p}}/JA_{\mathfrak{p}}).$$

Now use the multiplicity formula [**N**, p. 76]:

$$\begin{aligned}\delta(A) &= \textstyle\sum \delta(A/\mathfrak{p})\operatorname{length}(A_{\mathfrak{p}}) \quad (\text{sum over } \mathfrak{p} \in \operatorname{Ass}(A))\\ &= \textstyle\sum \delta(A/\mathfrak{p})\operatorname{length}(A_{\mathfrak{p}}/IA_{\mathfrak{p}}) + \sum \delta(A/\mathfrak{p})\operatorname{length}(A_{\mathfrak{p}}/JA_{\mathfrak{p}})\\ &= \delta(A/I) + \delta(A/J).\end{aligned}$$

(b) Since A is a 0-dimensional Gorenstein local ring, $\lambda(\operatorname{ann}(A_1)) = 1$. Hence $\operatorname{ann}(A_1) = A_N$. It follows easily that the k-bilinear map $A_i \times A_j \to A_{i+j}$, induced by multiplication, is nonsingular for $i + j \leqslant N$. From this one deduces that $(\operatorname{ann}(I_t))_{N-t} = J_{N-t}$. (b) now follows easily from the nonsingular k-bilinear pairing $A_t \times A_{N-t} \to A_N \cong k$.

Observation 1(b), a well-known property of quasi-Frobenius algebras, contains our theorem relating the Hilbert functions of linked projective schemes. To see this we require certain standard technicalities.

Further notation. $H(S, -)$ denotes the *Hilbert function* of the standard $\mathbf{N}$-graded k-algebra S (i.e., $H(S, t) = \lambda(S_t)$), and Δ denotes the difference operator on $\mathbf{Z}$-valued sequences (i.e., if f is a $\mathbf{Z}$-valued sequence, then $\Delta f(0) = f(0)$ and $\Delta f(i) = f(i) - f(i-1)$ for $i > 0$). Since $H(S, t)$ is a degree $\dim S - 1$ polynomial function of t for $t \gg 0$, $\Delta^{\dim S} H(S, t) = 0$ for $t \gg 0$. *Define*:

$$\sigma(S) = 1 + \max\{t \in \mathbf{N} | \Delta^{\dim S} H(S, t) \neq 0\}.$$

Observe that for any $t \geqslant \sigma(S) - 1$, $\delta(S) = \Sigma\{\Delta^{\dim S} H(S, j) | 0 \leqslant j \leqslant t\}$, and $\delta(S) = \Delta^{\dim S - 1} H(S, t)$ if $\dim S > 0$. For any nonzero, nonunit, homogeneous ideal Q of S, *define*:

$$\sigma(Q) = \sigma(S/Q); \quad \alpha(Q) = \min\{t \in \mathbf{N} | Q_t \neq 0\}.$$

Observe that for any $m \in \mathbf{N}$, $\alpha(Q) = \min\{t \in \mathbf{N} | \Delta^m H(S/Q, t) \neq \Delta^m H(S, t)\}$.

Reduction to dimension 0. Henceforth, for technical convenience, *we assume k to be infinite*, in which case there is an $A_1 A$-primary ideal Q generated by an A-regular sequence in A_1. (So this sequence is also (A/I)-regular and, if A/J is CM, then (A/J)-regular.) Let $x \mapsto \bar{x}$ denote the canonical map $A \to A/Q = \bar{A}$. By [**G**], $\lambda(\operatorname{ann}(\bar{A}_1))$ is independent of the choice of Q; this integer is called the *CM-type of A*. Recall that A is Gorenstein $\Leftrightarrow$ $\bar{A}$ is Gorenstein $\Leftrightarrow$ CM-type of $A = 1$.

2. OBSERVATIONS. Let $m = \dim A$.

(a) $\Delta^m H(A, -) = H(\bar{A}, -)$; $\Delta^m H(A/I, -) = H(\bar{A}/\bar{I}, -)$; $\bar{I} \neq (0)$.

(b) $\delta(A) = \delta(\bar{A}) = \lambda(\bar{A})$; $\delta(A/I) = \delta(\bar{A}/\bar{I}) = \lambda(\bar{A}/\bar{I})$.

(c) $\sigma(A) = \sigma(\bar{A})$; $\sigma(I) = \sigma(\bar{I})$; $\alpha(I) = \alpha(\bar{I})$.

(d) $\alpha(\bar{I}) \leqslant \sigma(\bar{I}) \leqslant \sigma(\bar{A}) \neq \alpha(\bar{I})$.

PROOF. (a) follows immediately from the fact that Q is generated by a sequence which is both A- and (A/I)-regular, and (b)–(d) follow formally from (a) and definitions.

3. THEOREM (HILBERT FUNCTIONS UNDER LIAISON). *Suppose A is Gorenstein. Let* $m = \dim A$, $N = \sigma(A) - 1$. *Then*:

(a) $\Delta^m H(A, t) = \Delta^m H(A, N - t), 0 \leqslant t \leqslant N$.

(b) $\Delta^m H(A, t) = \Delta^m H(A/I, t) + \Delta^m H(A/J, N - t), 0 \leqslant t \leqslant N$.

(c) $\alpha(I) + \sigma(J) = \alpha(J) + \sigma(I) = \sigma(A)$.

PROOF. First note that (c) is a formal consequence of (b) and definitions, and that (a) and (b) follow immediately from 1(b) if $m = 0$. Hence (a) and (b) follow immediately from 2(a, c), the 0-dimensional case, and

Claim. $\bar{J} = \operatorname{ann} \bar{I}$.

PROOF. By 2(b) and 1(a)

$$\lambda(\bar{A}/\bar{J}) = \delta(A/J) = \delta(A) - \delta(A/I) = \delta(\bar{A}) - \delta(\bar{A}/\bar{I})$$
$$= \delta(\bar{A}/\operatorname{ann} \bar{I}) = \lambda(\bar{A}/\operatorname{ann} \bar{I}).$$

Since $\bar{J} \subseteq \operatorname{ann} \bar{I}$, $\bar{J} = \operatorname{ann} \bar{I}$, and we are done.

REMARK. Suppose A is Gorenstein. Then, as a corollary to 3(c), we have the *validity of the Cayley-Bacharach theorem for A*:

$$\delta(A/I) = \delta(A) - 1 \Rightarrow \alpha(I) = \sigma(A) - 1.$$

(PROOF. $\delta(A/J) = 1$, whence $\sigma(J) = 1$.) We call this result "Cayley-Bacharach" because: specializing to the case in which $\operatorname{Proj}(A)$ is the complete intersection of two curves in $\mathbf{P}^2$, in which case $\sigma(A)$ is one less than the sum of the degrees of the curves, we obtain the classical Cayley-Bacharach theorem. (See [**SR**, pp. 97–101] for further details.) More generally, 3(c) gives: $\operatorname{Proj}(A/I)$ is in generic position in $\operatorname{Proj}(A) \Leftrightarrow \operatorname{Proj}(A/J)$ is in generic position in $\operatorname{Proj}(A)$. ("Generic position" simply means "$\sigma \leqslant \alpha + 1$"; see [**O**] for a geometric interpretation of "generic position in $\mathbf{P}^n$".) The validity of Cayley-Bacharach for A is, in fact, a consequence of only "half" of the Gorenstein property, since more generally we have

4. THEOREM. $\Delta^{\dim A} H(A, \sigma(A) - 1) \leqslant$ *CM-type of A. If equality holds, then*

$$\delta(A/I) \leqslant \delta(A) - \sigma(A) + \alpha(I) \leqslant \delta(A) - 1.$$

PROOF (CF. [**DM**, PROOF OF (2.3)]). CM-type of $A = \lambda(\operatorname{ann} \bar{A}_1) \geqslant \lambda(\bar{A}_N) = \Delta^{\dim A} H(A, N)$ $(N = \sigma(A) - 1)$; equality $\Leftrightarrow \operatorname{ann} \bar{A}_1 = \bar{A}_N$. If equality holds, then $\bar{I}_t \neq 0$ $(\alpha(\bar{I}) \leqslant t \leqslant N)$, whence, using 2(b, c, d):

$$\delta(A/I) = \lambda(\bar{A}/\bar{I}) \leqslant \lambda(\bar{A}) - (\sigma(\bar{A}) - \alpha(\bar{I}))$$
$$= \delta(A) - \sigma(A) + \alpha(I) \leqslant \delta(A) - 1.$$

REMARK. Observe that $\delta(A/I) = \delta(A) - 1 \Rightarrow J \in \operatorname{Ass}(A)$ and $\delta(A/J) = 1$, i.e., $\operatorname{Proj}(A/J)$ is a linear component of $\operatorname{Proj}(A)_{\mathrm{red}}$. The existence of such a component is guaranteed if and only if $\operatorname{Proj}(A)$ is 0-dimensional and has a k-rational point. That is, the natural domain of applicability of "Cayley-Bacharach" is that of 0-dimensional subschemes of $\mathbf{P}^n(k = \bar{k})$. Although such schemes may have "Cayley-Bacharach" without being arithmetically Gorenstein, we have

5. THEOREM. *Suppose that A is reduced*, $\dim A = 1$, *and every point of* $\operatorname{Proj}(A)$ *is k-rational. Then A is Gorenstein if and only if the following two conditions are satisfied.* (*Let* $N = \sigma(A) - 1$.)

(a) (*Symmetric Hilbert function*)

$$\Delta H(A, t) = \Delta H(A, N - t), \qquad 0 \leqslant t \leqslant N.$$

(b) (*Cayley-Bacharach*)

$$\alpha(\operatorname{ann} \mathfrak{p}) = N \quad \textit{for all } \mathfrak{p} \in \operatorname{Ass}(A).$$

PROOF. In view of 3, we need only prove the sufficiency of (a) and (b). Let C be the conductor of A in its integral closure B. We shall prove that $A_{A_1 A}$ is Gorenstein by verifying that $\lambda(B/C) = 2\lambda(A/C)$ [**HK**, p. 32].

Identify B as an A-algebra and an $\mathbf{N}$-graded k-algebra with $\oplus\{A/\mathfrak{p} | \mathfrak{p} \in \operatorname{Ass}(A)\}$. Note that $A/\mathfrak{p} \cong k[T]$ (graded k-algebra isomorphism). Under these circumstances C is the ideal in A (and in B), $\Sigma\{\operatorname{ann} \mathfrak{p} | \mathfrak{p} \in \operatorname{Ass}(A)\}$ [**O**, Proposition 2.5]. So, by (b), $C_t = 0$ for $0 \leqslant t < N$ and $\lambda(C_N) \geqslant \delta = \delta(A) = \operatorname{card}(\operatorname{Ass}(A))$. On the other hand, $\lambda(B_t) = \delta$ $(t \geqslant 0)$ and $\lambda(A_N) = H(A, N) = \delta$. Consequently, $A_N = B_N = C_N$, $\lambda(B/C) = N\delta$ and $\lambda(A/C) = \Sigma\{H(A, t) | 0 \leqslant t \leqslant N - 1\}$. Now, $H(A, t) = \Sigma\{\Delta H(A, j) | 0 \leqslant j \leqslant t\} = H(A, N) - \Sigma\{\Delta H(A, j) | t + 1 \leqslant j \leqslant N\}$. Then, using (a),

$$H(A, t) = \delta - \sum\{\Delta H(A, j) | 0 \leqslant j \leqslant N - 1 - t\} = \delta - H(A, N - 1 - t).$$

Consequently, $2\lambda(A/C) = 2(\Sigma\{H(A, t) | 0 \leqslant t \leqslant N - 1\}) = N\delta = \lambda(B/C)$.

REMARKS. We do not know to what extent the hypothesis "reduced" can be eliminated from 5. In case $\lambda(A_1) \leqslant 3$, i.e., in case $\operatorname{Proj}(A)$ is a subscheme of $\mathbf{P}^2$, [**DM**] proves 5 without "reduced", and a stronger result than 5 with "reduced". That analysis depends heavily on the fact that, in $\mathbf{P}^2$, "arithmetically Gorenstein" = "complete intersection". 5 should also be compared with Stanley's characterization of Gorenstein domains among the CM domains [**S**, Theorem 4.4].

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