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THE CONVEXITY OF A DOMAIN AND THE SUPERHARMONICITY OF THE SIGNED DISTANCE FUNCTION

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ABSTRACT. Let D be a domain in $\mathbf{R}^N$ with nonempty boundary ∂D and let u be the signed distance function from ∂D, i.e. $u = \pm$dist according as we are in or outside $\overline{D}$. We prove that, for any $N \geqslant 2$, u is superharmonic in $\mathbf{R}^N$ if and only if D is convex. When $N = 2$, this criterion requires the superharmonicity of u in D only.

1. Throughout this paper D will denote a proper subdomain of the Euclidean space $\mathbf{R}^N$, where $N \geqslant 2$. Thus the boundary ∂D of D in $\mathbf{R}^N$ is not empty and we can define the distance function d from ∂D. The *signed distance* function u in $\mathbf{R}^N$ is defined by

$$u = \begin{cases} d & \text{in } \overline{D}, \\ -d & \text{in } D', \end{cases}$$

where $\overline{D}$ is the closure of D in $\mathbf{R}^N$ and $D' = \mathbf{R}^N \setminus \overline{D}$.

Our main result is the following

THEOREM 1. *The function u is superharmonic in $\mathbf{R}^N$ if and only if D is convex.*

The "if" part of Theorem 1 must be known, at least tacitly; cf. Fuchs [**1**, p. 11]. For completeness we sketch a proof. For every support hyperplane H of $\overline{D}$ let u_H be the signed distance function from H such that $u_H > 0$ in D and u_H is harmonic in $\mathbf{R}^N$. Then $u = \inf_H u_H$ and it follows that u is superharmonic (in fact, u is concave) in $\mathbf{R}^N$, since the u_H are all harmonic and $u \in \mathscr{C}(\mathbf{R}^N)$. The proof of the "only if" part of Theorem 1 (cf. §3) is more involved and requires two preliminary lemmas (§2).

We note that, for example, if D is the punctured ball $D = \{X \in \mathbf{R}^N: 0 < r = \|X\| < 1\}$, then u is superharmonic in D' but not in D. With this motivation we now state

THEOREM 2. *If D is a planar domain and d is superharmonic in D, then D is convex. In higher dimensions, neither D nor $\overline{D}$ need be convex.*

THEOREM 3. *Let F be a proper closed subset of $\mathbf{R}^N$, where $N \geqslant 2$, and let d be the distance from ∂F. Then d is subharmonic in F' if and only if F is convex.*

The $N = 2$ and $N \geqslant 3$ cases of Theorem 2 are proved in §§4 and 5, Theorem 3 in §3.

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2. LEMMA 1. *Let $D \subset \mathbf{R}^N$ be such that $D \neq \operatorname{int}(\overline{D})$. Then u is not superharmonic in* $\mathbf{R}^N$.

We denote the mean-value of u on $S(X_0, r) = \{X: \|X - X_0\| = r\}$ by $M(u, X_0, r)$.

To prove Lemma 1, choose $X_0 \in \partial D$ and $r_0 > 0$ so that $B(X_0, r_0) = \{X: \|X - X_0\| < r_0\} \subset \operatorname{int}(\overline{D}) \subset \overline{D}$. Clearly $M(u, X_0, r) > 0$ if $0 < r < r_0$; thus if u were superharmonic we must have $u(X_0) > 0$.

LEMMA 2. *Let Y_1, Y_2 be distinct points in $\mathbf{R}^N$ such that $\|Y_1\| = \|Y_2\|$. Let r_1, r_2 denote the distances of a point from Y_1, Y_2, respectively, and define v in $\mathbf{R}^N$ by $v = r_1 \wedge r_2$. Then there exists a positive number r_0 such that $v(O) > M(v, O, r)$ for all r in $(0, r_0)$.*

By using a magnification, we may suppose that $\|Y_1\| = \|Y_2\| = 1$, and by rotating the axes, we may suppose further that $Y_1 = (\cos\phi, \sin\phi, 0, \ldots, 0)$ and $Y_2 = (-\cos\phi, \sin\phi, 0, \ldots, 0)$, where $0 \leqslant \phi < \pi/2$.

If $X = (x_1, \ldots, x_N) \in \mathbf{R}^N$ and $r = \|X\|$, then, writing

$$f(X) = r^2 - 2|x_1|\cos\phi - 2x_2\sin\phi,$$

we have

$$v(X) = (1 + f(X))^{1/2} \leqslant 1 + \tfrac{1}{2}f(X).$$

Hence

$$M(v, O, r) \leqslant 1 + \tfrac{1}{2}M(f, O, r) = 1 + \tfrac{1}{2}r^2 - (\cos\phi)M\big(|x_1|, O, r\big).$$

Since $M(|x_1|, O, r)$ is a positive multiple of r and $\cos\phi > 0$, we have $M(v, O, r) < 1 = v(O)$ when r is small.

3. To prove the "only if" in Theorem 1, suppose that D is not convex. If $\overline{D}$ is convex, then Lemma 1 implies that u is not superharmonic in $\mathbf{R}^N$, since then $\operatorname{int}(\overline{D})$ is convex [**2**, Theorem 1.11] and so $D \neq \operatorname{int}(\overline{D})$.

Now suppose that $\overline{D}$ is nonconvex. A key result for this case is Motzkin's theorem, which states that a proper closed subset F of $\mathbf{R}^N$ is convex if and only if each point of $\mathbf{R}^N$ has a unique nearest point of F (cf. [**2**, Theorem 7.8]). Hence, taking $F = \overline{D}$, we may assume that (by translating the origin, if necessary) $O \in D'$ and that there exist distinct points Y_1, Y_2 of $\overline{D}$ such that $d(O) = \|Y_1\| = \|Y_2\| > 0$. Define v in $\mathbf{R}^N$ by $v(X) = \|X - Y_1\| \wedge \|X - Y_2\|$. By Lemma 2, there exists $r_0 > 0$ such that $v(O) > M(v, O, r)$ whenever $0 < r < r_0$. Also, $\overline{B(O, r)} \subset D'$ for one of these r. Since $v(X) \geqslant d(X)$ for all X in D' with equality when $X = O$, we obtain

$$u(O) = -d(O) = -v(O) < -M(v, O, r) \leqslant -M(d, O, r) = M(u, O, r),$$

so that u is not superharmonic in D'.

The argument in the last paragraph (with $\overline{D}$ replaced by F) proves the "only if" in Theorem 3. The proof of "if" in Theorem 3 is similar to the proof of "if" in Theorem 1 (§1).

4. To prove the plane case ($N = 2$) of Theorem 2, we suppose that D is nonconvex in $\mathbf{R}^2$ and show that d is not superharmonic in D. There exist a point Y_0 of ∂D, a positive number ε and a closed half-plane P with Y_0 on ∂P such that

$$P \cap \big(\overline{B(Y_0, \varepsilon)} \setminus \{Y_0\}\big) \subset D;$$

cf. [**2**, Theorem 4.8]. Without loss of generality, suppose that $Y_0 = O$ and $P = \{X: x_2 \geqslant 0\}$. Let $X_0 = (0, \varepsilon/4)$ and $B = B(X_0, \varepsilon/8)$. If $X = (x_1, x_2) \in B$ and $x_1 \neq 0$, then $d(X) > x_2$. Hence, by the area mean-value equality for the function x_2,

$$\int_B d(X)\,dX > \int_B x_2\,dX = \pi(\varepsilon/8)^2(\varepsilon/4) = \pi(\varepsilon/8)^2\,d(X_0),$$

so that the area mean-value inequality for the superharmonicity of d fails at X_0.

5. Here we show by an example that in higher dimensions ($N \geqslant 3$) the superharmonicity of u in D does not necessarily imply the convexity of D, nor even of $\overline{D}$.

Let Ω denote the torus in $\mathbf{R}^3$ obtained by rotating the disc $\omega = \{(0, x_2, x_3): (x_2 - a)^2 + x_3^2 < 1\}$, where $a \geqslant 2$, about the x_3-axis. In the case $N = 3$ let $D = \Omega$, and in the case $N \geqslant 4$ let $D = \Omega \times \mathbf{R}^{N-3}$. Clearly D is not convex, and neither is $\overline{D}$. We shall show, however, that d is superharmonic in D.

With a point X (in $\mathbf{R}^N$) we associate plane polar coordinates (r, θ) such that $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ and we put $\rho = \rho(X) = (x_3^2 + (r - a)^2)^{1/2}$. Then $D = \{X: \rho < 1\}$ and $\partial D = \{X: \rho = 1\}$.

If $X \in D$, then, in finding $d(X)$, we may suppose that $(x_1, x_2, x_3) \in \omega$. Let $X_0 = (0, a, 0, \ldots, 0)$. Then $B(X, 1 - \|X - X_0\|) \subset B(X_0, 1) \subset D$ and so $d(X) \geqslant 1 - \|X - X_0\| = 1 - \rho$. If $X = X_0$, then clearly $d(X) = 1$; if $X \neq X_0$, then the point Y_0 such that $\|Y_0 - X_0\| = 1$ and X_0, X, Y_0 are collinear (in that order) belongs to ∂D, so that $d(X) \leqslant \|X - Y_0\| = 1 - \rho$. Hence, in all cases, $d(X) = 1 - \rho$.

Let $G = \{X: \rho = 0\}$. We show first that d is superharmonic in $D \setminus G$ by computing the Laplacian

$$\Delta d(X) = -\Delta\rho = -\left\{\frac{\partial^2\rho}{\partial x_3^2} + r^{-1}\frac{\partial\rho}{\partial r} + \frac{\partial^2\rho}{\partial r^2}\right\} = \frac{a - 2r}{r\rho};$$

as we have $2r > 2(a - 1) \geqslant a$, we get $\Delta d < 0$. Hence d is superharmonic in $D \setminus G$ and therefore satisfies the weak mean-value inequality in $D \setminus G$ (that is, if $S(X, r) \subset D \setminus G$, then $d(X) \geqslant M(d, X, r)$). Further, d takes its maximum value at each point of G and therefore the mean-value inequality holds on G, too. As d is continuous, it follows that d is superharmonic in D.

References

1. W. H. J. Fuchs, *Topics in the theory of functions of one complex variable*, Van Nostrand, Princeton, N. J., 1967.

2. F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

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