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A WEIGHTED WEAK TYPE INEQUALITY FOR THE MAXIMAL FUNCTION

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ABSTRACT. We show that the operator $S = v^{-1}Mv$, where M denotes the Hardy-Littlewood maximal operator, is of weak type (1,1) with respect to the measure $v(x)w(x)\,dx$ whenever v and w are A_1 weights. B. Muckenhoupt's weighted norm inequality for the maximal function can then be obtained directly from the P. Jones factorization of A_p weights using interpolation with change of measure.

In [**8**], B. Muckenhoupt characterized the nonnegative functions, or weights, w on R satisfying the weighted norm inequality ($1 < p < \infty$)

$$\int_{-\infty}^{\infty} |Mf(x)|^p w(x)\,dx \leqslant C\int_{-\infty}^{\infty} |f(x)|^p w(x)\,dx \quad \text{for all } f, \tag{1}$$

where $Mf(x) = (\sup_{x\in I} 1/|I|)\int_I |f(y)|\,dy$ is the Hardy-Littlewood maximal function of f, as those weights w satisfying the A_p condition

$$\left[\frac{1}{|I|}\int_I w\right]\left[\frac{1}{|I|}\int_I w^{1-p'}\right]^{p-1} \leqslant C \quad \text{for all intervals } I. \tag{A_p}$$

Later, P. Jones showed in [**7**] that a weight w satisfies the A_p condition if and only if it admits a factorization

$$w = w_0 w_1^{1-p} \text{ where } Mw_j(x) \leqslant Cw_j(x) \text{ for all } x,\ j = 0,1. \tag{2}$$

More recently, M. Christ and R. Fefferman [**2**] have given an elementary proof of the implication $(A_p) \Rightarrow (1)$ (see also R. Hunt, D. Kurtz and C. Neugebauer [**5**], B. Jawerth [**6**] and E. Sawyer [**10**]), and R. Coifman, P. Jones and J. Rubio de Francia [**4**] have given a short proof that (2) follows from (1) and its "dual" inequality, the boundedness of M on $L^{p'}(w^{1-p'})$.

A natural approach to obtaining inequality (1) directly from the factorization in (2) is provided by the Stein-Weiss interpolation with change of measures theorem [**11**]. In order to see what is needed, suppose (2) holds and, following the proof in [**11**], define $Sf = w_1^{-1}M(w_1 f)$. Note that (1) can be rewritten

$$\int |Sf|^p w_0 w_1 \leqslant C\int |f|^p w_0 w_1 \quad \text{for all } f. \tag{3}$$

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Now S is bounded on $L^\infty(w_0 w_1)$ simply because $Mw_1 \le Cw_1$ and thus (3) will follow from the usual Marcinkiewicz interpolation theorem provided that S is of weak type $(1,1)$ with respect to the measure $w_0(x)w_1(x)\,dx$.

THEOREM. *Suppose v and w satisfy the A_1 condition, i.e. $Mv(x) \leqslant Av(x)$ and $Mw(x) \leqslant Bw(x)$ for all x. Then*

$$(4) \qquad \int_{\{Mg > v\}} v(x)w(x)\,dx \leqslant C\int_{-\infty}^{\infty} g(x)w(x)\,dx \quad \textit{for all } g \geqslant 0 \textit{ on } R,$$

where the constant C depends only on A and B. This shows (with $g = \lambda^{-1}fv$) that the operator $Sf = v^{-1}M(vf)$ is of weak type $(1,1)$ with respect to $v(x)w(x)\,dx$.

It would be of interest to obtain an analogue of the above approach for two weight inequalities and other operators, specifically the Hilbert transform. Regarding earlier work and other weighted weak type inequalities for the maximal function, see K. Andersen and B. Muckenhoupt [**1**], B. Muckenhoupt [**8**] and especially the treatment given by B. Muckenhoupt and R. L. Wheeden in [**9**] from which our proof borrows heavily. The letter C will denote a positive constant that may change from line to line and $|E|_v = \int_E v(x)\,dx$, $|E| = \int_E dx$ for $v \geqslant 0$ on R, $E \subset R$.

PROOF. It suffices to prove (4) for $g \geqslant 0$ bounded with compact support. Fix such a g. For $k \in Z$, let $\{I_j^k\}_j$ be the collection of component intervals of the open set $\Omega_k = \{Mv > 3^k\} \cap \{Mg > 3^k\}$. Denote by Γ the set of pairs (k, j) such that $I_j^k \cap \{v \leqslant 3^{k+1}\}$ has positive measure. For $(k,j) \in \Gamma$ we then have

$$(5) \qquad \begin{aligned} \frac{3^k}{A} &\leqslant A^{-1} \operatorname*{ess\,inf}_{I_j^k} Mv \leqslant \operatorname*{ess\,inf}_{I_j^k} v \leqslant \left|I_j^k\right|^{-1} \int_{I_j^k} v \\ &\leqslant \operatorname*{ess\,inf}_{I_j^k} Mv \leqslant A \operatorname*{ess\,inf}_{I_j^k} v \leqslant 3^{k+1}A, \end{aligned}$$

since $Mv \leqslant Av$. Thus

$$(6) \qquad \begin{aligned} \int_{\{Mg>v\}} vw &\leqslant 3\sum_k 3^k \left|\{3^k < v \leqslant 3^{k+1}\} \cap \{Mg > v\}\right|_w \\ &\leqslant 3\sum_k 3^k \sum_{j:\,(k,j)\in\Gamma} \left|I_j^k\right|_w \\ &\leqslant 3A \sum_{(k,j)\in\Gamma} \left|I_j^k\right|^{-1} \left|I_j^k\right|_v \left|I_j^k\right|_w \quad \text{by (5).} \end{aligned}$$

For $N \in Z$, set $\Gamma_N = \{(k,j) \in \Gamma : k \geqslant N\}$. We shall prove

$$(7) \qquad \sum_{(k,j)\in\Gamma_N} \left|I_j^k\right|^{-1} \left|I_j^k\right|_v \left|I_j^k\right|_w \leqslant C\int gw \quad \text{for all } N$$

with a constant C independent of N. The proof uses a variant of an idea of B. Muckenhoupt and R. L. Wheeden [**9**; Proof of Lemma 2]. First note that for (k,j), $(t,s) \in \Gamma_N$ with $k \geqslant t$, either $I_j^k \subset I_s^t$ or $I_j^k \cap I_s^t = \varnothing$. Fix N and let G_0 consist of the indices $(k,j) \in \Gamma_N$ for which I_j^k is maximal in $\{I_s^t : (t,s) \in \Gamma_N\}$. Since $Mv \leqslant Av$, v satisfies the A_∞ condition [**3**] and thus there are positive constants C, ε such that

$$(8) \qquad |E|_v/|I|_v \leqslant C\big(|E|/|I|\big)^{\varepsilon} \quad \text{whenever } E \text{ is a subset of an interval } I.$$

Choose $0 < \delta < \varepsilon$. If G_n has been defined, let G_{n+1} consist of those $(k, j) \in \Gamma$ for which there is $(t, s) \in G_n$ with $I_j^k \subset I_s^t$ and

$$(9)\quad \begin{aligned} &\text{(i)} \quad \frac{1}{|I_j^k|}\int_{I_j^k} w > 3^{(k-t)\delta}\frac{1}{|I_s^t|}\int_{I_s^t} w, \\ &\text{(ii)} \quad \frac{1}{|I_i^l|}\int_{I_i^l} w \leqslant 3^{(l-t)\delta}\frac{1}{|I_s^t|}\int_{I_s^t} w \quad \text{whenever } (l, i) \in \Gamma \text{ and } I_j^k \subsetneq I_i^l \subset I_s^t. \end{aligned}$$

Let $P = \bigcup_{n=0}^{\infty} G_n$. Following **[9]**, we claim that

$$(10)\qquad \sum_{(k,j)\in\Gamma_N} |I_j^k|^{-1}|I_j^k|_v|I_j^k|_w \leqslant C \sum_{(k,j)\in P} |I_j^k|^{-1}|I_j^k|_v|I_j^k|_w.$$

To see this, suppose $(t, s) \in P$ and let $Q = Q(t, s)$ denote the set of indices $(k, j) \in \Gamma$ such that $I_j^k \subset I_s^t$ and there is no $(l, i) \in P$ with $I_j^k \subset I_i^l \subsetneq I_s^t$. Then by (9)(ii)

$$\begin{aligned} \sum_{(k,j)\in Q} |I_j^k|^{-1}|I_j^k|_w|I_j^k|_v &\leqslant \sum_{(k,j)\in Q} 3^{(k-t)\delta}|I_s^t|^{-1}|I_s^t|_w|I_j^k|_v \\ &\leqslant |I_s^t|^{-1}|I_s^t|_w \sum_{k=t}^{\infty} 3^{(k-t)\delta}|I_s^t|_v C\left(\frac{|\{Mv > 3^k\} \cap I_s^t|}{|I_s^t|}\right)^{\varepsilon} \quad \text{by (8)}. \end{aligned}$$

However, $Mv \leqslant Av$ and so $|\{Mv > 3^k\} \cap I_s^t| \leqslant A3^{-k}|I_s^t|_v \leqslant A^2 3^{t-k+1}|I_s^t|$ by (5) and thus the final line above is dominated by

$$\begin{aligned} &CA^{2\varepsilon}|I_s^t|^{-1}|I_s^t|_w|I_s^t|_v \sum_{k=t}^{\infty} 3^{(k-t)\delta}3^{(t-k+1)\varepsilon} \\ &\leqslant C|I_s^t|^{-1}|I_s^t|_w|I_s^t|_v \quad \text{since } 0 < \delta < \varepsilon. \end{aligned}$$

Inequality (10) now follows since $\bigcup_{(t,s)\in P} Q(t, s) = \Gamma_N$.

For each k in Z, let $\{J_i^k\}_i$ be the component intervals of $\{Mg > 3^k\}$. Then $M(\chi_{J_i^k} g) > 3^k$ on J_i^k and so

$$(11)\qquad |J_i^k| \leqslant \left|\{M(\chi_{J_i^k} g) > 3^k\}\right| \leqslant C3^{-k}\int_{J_i^k} g,$$

since the maximal operator is of weak type (1, 1) with respect to Lebesgue measure. Given an interval I_j^k, there is a unique $i = i(k, j)$ such that $I_j^k \subset J_i^k$. From now on, whenever the index i appears in a summation over (k, j), it is understood that $i = i(k, j)$. We have

$$(12)\qquad \begin{aligned} \sum_{(k,j)\in P} |I_j^k|^{-1}|I_j^k|_v|I_j^k|_w &\leqslant 3A \sum_{(k,j)\in P} 3^k|I_j^k|_w \quad \text{by (5)} \\ &\leqslant CA \sum_{(k,j)\in P} \frac{1}{|J_i^k|}\left(\int_{J_i^k} g\right)|I_j^k|_w \quad \text{by (11)} \\ &= CA\int\left[\sum_{(k,j)\in P} |J_i^k|^{-1}|I_j^k|_w \chi_{J_i^k}\right] g. \end{aligned}$$

Let $h(x) = \Sigma_{(k,j)\in P}|J_i^k|^{-1}|I_j^k|_w \chi_{J_i^k}(x)$. It remains to show that $h(x) \leqslant Cw(x)$ for all $x \in R$. So fix $x \in R$. For any given k, there is at most one interval J_i^k containing x. We denote this interval, when it exists, by J^k. Let $P_k = \{(k, j) \in P\colon I_j^k \subset J^k\}$ and let $G = \{k\colon P_k \neq \varnothing\}$. Let k_0 be the least integer k in G and if $k_0, k_1, \ldots, k_n$ have been defined, choose k_{n+1} in G such that $k_{n+1} > k_n$ and

$$\text{(13)}\qquad \begin{aligned} &\text{(i)} && \frac{1}{|J^{k_{n+1}}|}\int_{J^{k_{n+1}}} w > 2\frac{1}{|J^{k_n}|}\int_{J^{k_n}} w, \\ &\text{(ii)} && \frac{1}{|J^l|}\int_{J^l} w \leqslant 2\frac{1}{|J^{k_n}|}\int_{J^{k_n}} w \quad \text{for } k_n \leqslant l < k_{n+1}, l \in G. \end{aligned}$$

We now claim that

$$\text{(14)}\qquad \sum_{\substack{l\in G\\ k_n \leqslant l < k_{n+1}}} \sum_{(l,j)\in P_l} \frac{|I_j^l|_w}{|J^l|_w} \leqslant C \quad \text{for } n \geqslant 0.$$

First, we observe that if $(l, j) \in P_l$, $k_n \leqslant l < k_{n+1}$, then

$$\text{(15)}\qquad \frac{1}{|I_j^l|}\int_{I_j^l} w > \frac{3^{(l-k_n)\delta}}{2B}\frac{1}{|J^l|}\int_{J^l} w.$$

To see this, let $I_s^{k_n}$ be the component of Ω_{k_n} that contains I_j^l. We claim that $(k_n, s) \in \Gamma$. Since $P \subset \Gamma$, it suffices to consider the case $(k_n, s) \notin P$. Since $k_n \in G$, J^{k_n} must contain at least one interval of the form $I_u^{k_n}$ with $(k_n, u) \in P$ and it follows that $J^{k_n} \supsetneqq I_s^{k_n}$. By the definition of Ω_{k_n}, we must have $Mv \leqslant 3^{k_n}$ at one of the endpoints of $I_s^{k_n}$. Thus the average of v over $I_s^{k_n}$ is at most 3^{k_n} and so

$$|\{v \leqslant 3^{k_n+1}\} \cap I_s^{k_n}| > 0.$$

Hence $(k_n, s) \in \Gamma$ by definition. Now let I_σ^k denote the smallest interval containing $I_s^{k_n}$ with $(k, \sigma) \in P$. Sufficiently many applications of (9)(i) yield

$$\frac{1}{|I_j^l|}\int_{I_j^l} w > 3^{(l-k)\delta}\frac{1}{|I_\sigma^k|}\int_{I_\sigma^k} w$$

and, since $(k_n, s) \in \Gamma$, (9)(ii) shows that

$$\frac{1}{|I_s^{k_n}|}\int_{I_s^{k_n}} w \leqslant 3^{(k_n-k)\delta}\frac{1}{|I_\sigma^k|}\int_{I_\sigma^k} w.$$

Finally, from (13)(ii) we have

$$\frac{1}{|J^l|}\int_{J^l} w \leqslant \frac{2}{|J^{k_n}|}\int_{J^{k_n}} w \leqslant 2B \operatorname*{ess\,inf}_{J^{k_n}} w \leqslant \frac{2B}{|I_s^{k_n}|}\int_{I_s^{k_n}} w$$

and, combining this with the two previous inequalities, we obtain (15). From (15) and the assumption $Mw \leqslant Bw$ we obtain that for $k_n \leqslant l < k_{n+1}$

$$\text{(16)}\qquad \bigcup_{j\colon (l,j)\in P_l} I_j^l \subset \left\{ w > \frac{3^{(l-k_n)\delta}|J^l|_w}{2B^2|J^l|} \right\} \cap J^l.$$

Since w satisfies the A_∞ condition [3], there are positive constants C, η such that $|E|_w/|I|_w \leqslant C(|E|/|I|)^\eta$ whenever $E \subset$ an interval I. Taking for E the set on the right side of (16), we conclude from the A_∞ condition and the inequality

$$|\{w > \lambda\} \cap J^l| \leqslant \lambda^{-1}|J^l|_w,$$

that $|E|_w$ is dominated by $|J^l|_w$ times $C(2B^2/3^{(l-k_n)\delta})^\eta$. It follows that the left side of (14) is dominated by $\sum_{l=k_n}^\infty C3^{(k_n-l)\delta\eta} \leqslant C$, as required.

We can now complete the proof. We have

$$\begin{aligned} h(x) &= \sum_{(k,j)\in P} \frac{|I_j^k|_w}{|J_i^k|_w}\left(\frac{1}{|J_i^k|}\int_{J_i^k} w\right)\chi_{J_i^k}(x) \\ &= \sum_n \sum_{\substack{l\in G \\ k_n\leqslant l<k_{n+1}}} \left(\sum_{(l,j)\in P_l} \frac{|I_j^l|_w}{|J^l|_w}\right)\left(\frac{1}{|J|^l}\int_{J^l} w\right) \\ &\leqslant \sum_n 2C\left(\frac{1}{|J^{k_n}|}\int_{J^{k_n}} w\right) \end{aligned}$$

by (13)(ii) and (14). By (13)(i), this last sum is dominated by twice its largest term which in turn is dominated by $CMw(x) \leqslant CBw(x)$. Thus $h(x) \leqslant Cw(x)$ and, combining this with (10) and (12), we obtain (7). Letting $N \downarrow -\infty$ and using (6), we obtain (4).

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