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CRITERIA FOR A BLASCHKE QUOTIENT TO BE OF UNIFORMLY BOUNDED CHARACTERISTIC

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ABSTRACT. Criteria for a quotient B_1/B_2 of Blaschke products B_1 and B_2 to be of uniformly bounded characteristic are proposed in terms of interpolating sequences.

A Blaschke product is a holomorphic function

$$B(z; \{a_n\}) = \prod_{n=1}^{\infty} \left(\frac{|a_n|}{a_n} \right) \frac{a_n - z}{1 - \bar{a}_n z}$$

in the disk $D = \{|z| < 1\}$, where $\{a_n\}$ is a sequence of complex numbers in D with $\Sigma(1 - |a_n|) < \infty$, with the convention $|a_n|/a_n = 1$ for $a_n = 0$. A Blaschke quotient is a meromorphic function B_1/B_2, where B_1 and B_2 are Blaschke products with no common zero. J. A. Cima and P. Colwell [**2**, Theorem 2] established a criterion for B_1/B_2 to be normal in D in the sense of O. Lehto and K. I. Virtanen [**4**] in terms of interpolating sequence in the sense of L. Carleson [**1**]. Here a function f meromorphic in D is normal if $(1 - |z|^2)f^{\#}(z)$ is bounded where, $f^{\#} = |f'|/(1 + |f|^2)$, and a sequence of points $\{z_n\}$ in D is interpolating (or uniformly separated [**3**, p. 148]) in D if

$$\inf_{n \geqslant 1} \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| > 0.$$

If $\{z_n\}$ is interpolating in D, then $\Sigma(1 - |z_n|) < \infty$. Cima and Colwell's cited result is (I) $\Leftrightarrow$ (II) in

THEOREM. *Let $\{a_n^{(1)}\}$ and $\{a_n^{(2)}\}$ be disjoint interpolating sequences of points in D, and set*

$$B_k(z) = B\left(z; \{a_n^{(k)}\}\right), \qquad k = 1, 2.$$

Then the following are mutually equivalent.

(I) *B_1/B_2 is normal in D.*

(II) *The sequence $\{a_n^{(1)}\} \cup \{a_n^{(2)}\}$ is interpolating in D.*

(III) *B_1/B_2 is of uniformly bounded characteristic in D.*

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To explain the terminology we set $f_w(z) = f((z+w)/(1+\bar{w}z))$, $z, w \in D$, for f meromorphic in D. We call f, of uniformly bounded characteristic in D, $f \in$ UBC for short, if

$$T(1, f_w) = \lim_{r \to 1-0} T(r, f_w), \qquad w \in D,$$

is bounded in D, where

$$T(r, f_w) = \int_0^r (\pi t)^{-1} \left[\iint_{|z|<t} (f_w)^{\#}(z)^2 \, dx\, dy \right] dt$$

is the Shimizu-Ahlfors characteristic function of f_w; see [**5**]. Since $f = f_0$, each $f \in$ UBC is of bounded (Nevanlinna) characteristic in D.

Since each $f \in$ UBC is normal in D by [**5**, Theorem 3.1, p. 383], (III) $\Rightarrow$ (I) is obvious. To establish our Theorem the remaining work is

PROOF OF (II) $\Rightarrow$ (III). First we observe, for $f = B_1/B_2$, the identity

$$T(1, f_w) = F^{\hat{}}(w) - F(w), \qquad w \in D, \tag{1}$$

where $F = \frac{1}{2}\log(|B_1|^2 + |B_2|^2)$ and $F^{\hat{}}$ is the least harmonic majorant of the subharmonic function F in D. For the proof of (1) we fix $w \in D$, and we let

$$D(w, r) = \{\zeta;\, |(\zeta - w)/(1 - \bar{w}\zeta)| < r\}, \qquad 0 < r < 1.$$

Then the Green function of $D(w, r)$ with its pole at w is

$$g_r(\zeta) = \log|r(1 - \bar{w}\zeta)/(\zeta - w)|, \qquad \zeta \in D(w, r).$$

Since the Laplacian $\Delta F = 2f^{\#2}$ in D, the change of variable $z = (\zeta - w)/(1 - \bar{w}\zeta)$, $\zeta \in D(w, r)$, together with

$$T(r, f_w) = \pi^{-1} \iint_{|z|<r} (f_w)^{\#}(z)^2 \log|r/z| \, dx\, dy,$$

derived from [**5**, (2.5), p. 352] for f_w, yields

$$T(r, f_w) = (2\pi)^{-1} \iint_{D(w,r)} (\Delta F(\zeta)) g_r(\zeta)\, d\xi\, d\eta.$$

The Green formula

$$\iint_G (\phi \Delta \psi - \psi \Delta \phi)\, dx\, dy = \int_{\partial G} \left(\psi \frac{\partial \phi}{\partial \nu} - \phi \frac{\partial \psi}{\partial \nu} \right) ds$$

for $G = D(w, r) \setminus D(w, \varepsilon)$ $(0 < \varepsilon < r)$, $\phi = g_r$, $\psi = (2\pi)^{-1}F$, in the limiting case $\varepsilon \downarrow 0$, then reads

$$T(r, f_w) = F_r^{\hat{}}(w) - F(w), \tag{2}$$

where $F_r^{\hat{}}$ is the least harmonic majorant of F in $D(w, r)$; to be more precise,

$$F_r^{\hat{}}(w) = (2\pi)^{-1} \int_{\partial D(w,r)} F(\zeta) \frac{\partial}{\partial \nu} g_r(\zeta)\, ds,$$

where $\partial/\partial\nu$ denotes the derivative in the inward-normal direction and ds is the element of arc length. Letting $r \uparrow 1$ in (2) we obtain (1).

Suppose (II), and suppose then that

$$\inf_{z \in D} F(z) = -\infty.$$

Then there exists a sequence $\{z_n\}$ in D such that

$$|B_1(z_n)|^2 + |B_2(z_n)|^2 \to 0.$$

By the same argument as in [**2**, p. 798], this contradicts our hypothesis (II). Therefore F is bounded from below and above in D: $-\infty < m \leqslant F \leqslant \log\sqrt{2}$. Thus,

$$F^{\hat{}} - F \leqslant \log\sqrt{2} - m \quad \text{in } D,$$

whence $f \in$ UBC by (1).

REMARK. It follows from (II) that

$$\text{(IV)} \qquad \inf_{z \in D} \left(|B_1(z)|^2 + |B_2(z)|^2 \right) > 0.$$

Our proof shows that (IV) $\Rightarrow$ (III).

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