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ON FIXED POINTS OF LINEAR CONTRACTIONS

HEYDAR RADJAVI AND PETER ROSENTHAL

ABSTRACT. It is shown that a weakly closed convex semigroup of linear contractions on a separable Hilbert space has a common fixed point other than 0 if the operator 0 is not in the semigroup.

We prove a theorem on existence of common fixed points for certain convex semigroups of linear operators on Banach spaces. The special case where the semigroup is a group follows easily from Kakutani's well-known theorem [**4**, **5**, **6**] and also, as discussions with P. Milman revealed, from the work of Brodskii and Milman [**1**]. Similarly, in the case where the semigroup is commutative, our result is a corollary of a special case of the Markov-Kakutani theorem [**3**, **4**]. Nonetheless, it appears that the results and corollaries given below have not been noticed before. Corollary 4, for example, gives a sufficient condition that $\bigvee_{n=N}^{\infty}\{A^n\}$ be the same for all N.

The applications of the fixed-point theorem that we consider concern operators on Hilbert space, but it seems worthwhile to state the theorem more generally.

THEOREM 1. *Let $\mathscr{X}$ be a strictly convex reflexive Banach space, and let $\mathscr{S}$ be a weak operator closed separable convex semigroup of linear contractions on $\mathscr{X}$. Then the operators in $\mathscr{S}$ have a common fixed point other than* 0 *if and only if the operator* 0 *is not in $\mathscr{S}$.*

PROOF. Clearly, if the operator 0 is in $\mathscr{S}$, then the only common fixed point is 0.

To prove the converse first recall that $(T_\alpha) \to T$ in the weak operator topology if and only if $\phi(T_\alpha x) \to \phi(Tx)$, for each $\phi \in \mathscr{X}^*$ and $x \in \mathscr{X}$. We require the fact that the unit ball of $\mathscr{B}(\mathscr{X})$ is weak operator compact; this can be proven as in the better-known case of Hilbert space. (That is, consider the Cartesian product of the closed balls of radius $\|x\|$ in $\mathscr{X}$, indexed by $\mathscr{X}$, where each ball is given the weak topology).

Let $\{T_n\}_{n=1}^{\infty}$ be a countable weak operator dense subset of $\mathscr{S}$; it obviously suffices to find a common fixed point for the $\{T_n\}$. Let

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n;$$

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this series converges in the norm topology (hence also in the weak operator topology) of $\mathscr{B}(\mathscr{X})$, and the closed convexity of $\mathscr{S}$ implies $T \in \mathscr{S}$. Now T defines a mapping of $\mathscr{S}$ into itself by $T(S) = TS$ for $S \in \mathscr{S}$ ($\mathscr{S}$ is a semigroup). Since $\mathscr{S}$ is a compact convex set, Schauder's fixed point theorem yields an operator $S_0 \in \mathscr{S}$ such that $TS_0 = S_0$. Choose $x \in \mathscr{X}$ such that $S_0 x \neq 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} T_n S_0 x = S_0 x.$$

For each n_0,

$$\left\| \sum_{n \neq n_0} \frac{1}{2^n} T_n S_0 x + \frac{1}{2^{n_0}} T_{n_0} S_0 x \right\| = \|S_0 x\|,$$

$$\left\| \sum_{n \neq n_0} \frac{1}{2^n} T_n S_0 x \right\| \leqslant \left(\sum_{n \neq n_0} \frac{1}{2^n} \right) \|S_0 x\|,$$

and

$$\left\| \frac{1}{2^{n_0}} T_{n_0} S_0 x \right\| \leqslant \frac{1}{2^{n_0}} \|S_0 x\|$$

imply that the above inequalities are equations, so the strict convexity of $\mathscr{X}$ implies that $T_{n_0} S_0 x$ is a multiple of $\sum_{n \neq n_0} T_n S_0 x / 2^n$. Hence, $T_{n_0} S_0 x$ is a multiple of $S_0 x$. (Recall that $\mathscr{X}$ strictly convex means that $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$ implies $\{x_1, x_2\}$ is linearly dependent). Thus, $T_n S_0 x$ is a multiple of $S_0 x$ for every n. But $\{\lambda_n\}$, complex numbers, satisfying $\sum_{n=1}^{\infty} \lambda_n / 2^n = 1$ and $|\lambda_n| \leqslant 1$ for all n implies $\lambda_n = 1$ for all n, so $T_n S_0 x = S_0 x$ for all n. Therefore, $S_0 x$ is a common fixed point for $\{T_n\}$ and, hence, for $\mathscr{S}$.

REMARK. As the referee has kindly pointed out, the above proof is similar to a proof given by R. E. Bruck, Jr., *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.

COROLLARY 1. *A weakly closed convex semigroup of contractions on a separable Hilbert space has a common fixed point other than* 0 *if and only if it does not contain the operator* 0.

PROOF. A Hilbert space satisfies all the hypotheses on $\mathscr{X}$ in Theorem 1. Also, the unit ball of operators on a separable Hilbert space is a separable metrizable space in the weak operator topology, so every semigroup of contractions is separable.

For the next two corollaries let $\mathscr{S}$ be a weakly closed convex semigroup of contractions on a separable Hilbert space.

COROLLARY 2. *Let $\mathscr{M}$ denote the set of common fixed points of members of $\mathscr{S}$; then $\mathscr{S}$ contains the orthogonal projection onto $\mathscr{M}$.*

PROOF. As is well known, $\|T\| \leqslant 1$ and $Tx = x$ implies $T^*x = x$ (begin an orthonormal basis with $x/\|x\|$ and represent T with respect to it). Thus, $\mathscr{M}$ reduces every operator in $\mathscr{S}$. Now $\mathscr{S}|\mathscr{M}^\perp$ is a weakly closed convex semigroup of contractions on $\mathscr{M}^\perp$. Since the only common fixed point of $\mathscr{S}|\mathscr{M}^\perp$ is $\{0\}$, Corollary 1 implies that the 0 operator is in $\mathcal{S}|\mathcal{M}^\perp$. Let $P \in \mathcal{S}$ be such that $P|\mathcal{M}^\perp = 0$; since $P|\mathcal{M}$ is the identity, P is the projection on $\mathcal{M}$.

COROLLARY 3. *If $\mathcal{S}$ is not the semigroup consisting only of the identity, then some operator in $\mathcal{S}$ has nontrivial nullspace.*

PROOF. By Corollary 2, if no operator in $\mathcal{S}$ has nullspace, then the set of common fixed points is the entire space.

The next result is a corollary of Theorem 1 in some cases but not in all. The proof, however, is contained in that of Theorem 1.

THEOREM 2. *If $\mathcal{S}$ is a weak operator closed bounded convex set of linear operators on a reflexive space and $0 \notin \mathcal{S}$, then 1 is an eigenvalue of every operator T with the property that $S \in \mathcal{S}$ implies $TS \in \mathcal{S}$.*

PROOF. Let T be as stated. By Schauder's theorem, $TS_0 = S_0$ for some $S_0 \in \mathcal{S}$. Choose x such that $S_0 x \neq 0$; then $TS_0 x = S_0 x$, so 1 is an eigenvalue of T.

COROLLARY 4. *If A is an injective operator on Hilbert space, and if there is a k such that $\|(1 + A)^n\| \leqslant k$ for every positive integer n, then the weakly closed linear span of $\{A^n: n \geqslant N\}$ is the same for all nonnegative integers N.*

PROOF. Let $T = 1 + A$ and let $\mathcal{S}$ be the weakly closed convex hull of $\{T^n: n \geqslant 1\}$. Since A has no nullspace, T has no fixed points other than 0. By Theorem 2, $0 \in \mathcal{S}$. Thus, given any weak operator neighborhood $\mathcal{W}$ of 0 there is a collection of nonnegative numbers $\{\lambda_j\}_{j=1}^m$ such that $\sum_{j=1}^m \lambda_j = 1$ and $\sum_{j=1}^m \lambda_j T^j \in \mathcal{W}$. Then $\sum_{j=1}^m \lambda_j T^j$ has the form $1 + \sum_{j=1}^m \lambda_j p_j(A)$ for suitable polynomials p_j without constant terms. It follows that 1 is in the weak closure of the linear span of $\{A^n: n \geqslant 1\}$. Thus, A is also in the weak closure of the linear span of $\{A^n: n \geqslant 2\}$ (multiplication is separately weakly continuous in each variable), and the corollary follows by a trivial induction.

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DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX B3H 4H8, NOVA SCOTIA, CANADA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO M5S 1A7, ONTARIO, CANADA