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SELFADJOINTNESS OF *-REPRESENTATIONS GENERATED BY POSITIVE LINEAR FUNCTIONALS

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ABSTRACT. The first purpose of this paper is to prove that π_τ is selfadjoint when π_ϕ is selfadjoint and π_ψ is bounded, where τ is the sum of positive linear functionals ϕ, ψ on a *-algebra $\mathscr{A}$ and π_τ, π_ϕ and π_ψ are *-representations generated by τ, ϕ and ψ, respectively. The second purpose is to prove that π_ϕ is standard, where ϕ is a positive linear functional on $\mathscr{A}$ such that there exists a net $\{\phi_\alpha\}$ of positive linear functionals on $\mathscr{A}$ satisfying $\phi_\alpha \leqq \phi$, π_{ϕ_α} is bounded for all α and $\lim_\alpha \phi_\alpha(x) = \phi(x)$ for each $x \in \mathscr{A}$.

1. Introduction. Unbounded operator algebras are important in connection with quantum field theory, the representation of Lie algebras and so on. This fact has led a number of mathematicians to start studying algebras of unbounded operators (Borchers, Uhlmann, Lassner, Powers, Schmüdgen, Gudder, etc.). In particular, Powers introduced the notions of closed hermitian and selfadjoint representations in analogy with the notions of closed, hermitian and selfadjoint operators, respectively; and further he introduced the notion of standard representation which is stronger than that of selfadjoint representation. The notions of selfadjointness and standardness have been indispensable in order to study unbounded representations in detail. However, since one difficulty lies in the judgement of selfadjointness and standardness, it seems worthwhile to study these questions. From this view point, we have attacked this problem [**4**, **5**, **9**].

In this paper we obtain some results with respect to the selfadjointness and standardness of *-representations generated by positive linear functionals.

Let ϕ be a positive linear functional on a *-algebra $\mathscr{A}$ with identity e. The well-known GNS-construction yields a triple $(\pi_\phi, \lambda_\phi, \mathfrak{H}_\phi)$, where π_ϕ is a closed *-representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}_\phi$, and λ_ϕ is a linear map of $\mathscr{A}$ into the domain $\mathscr{D}(\pi_\phi)$ of π_ϕ satisfying: $\lambda_\phi(\mathscr{A})$ is dense in $\mathscr{D}(\pi_\phi)$ with respect to the induced topology t_{π_ϕ}, and $\lambda_\phi(xy) = \pi_\phi(x)\lambda_\phi(y)$ for all $x, y \in \mathscr{A}$ [**7**]. Let ϕ and ψ be positive linear functionals on $\mathscr{A}$ and $\tau = \phi + \psi$. As an analogy of the well-known fact in the operator theory, we obtain that if π_ϕ is selfadjoint (resp. standard) and π_ϕ is bounded, then π_τ is selfadjoint (resp. standard). Further, we obtain that π_τ is selfadjoint if and only if π_τ is standard when π_ϕ and π_ψ are standard.

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We next consider a positive linear functional ϕ on $\mathscr{A}$ such that there exists a net $\{\phi_\alpha\}$ of positive linear functionals on $\mathscr{A}$ satisfying $\phi_\alpha \leqslant \phi$ for each α and $\lim_\alpha \phi_\alpha(x) = \phi(x)$ for each $x \in \mathscr{A}$. Suppose π_{ϕ_α} is bounded for all α. Takesue proved in **[9]** that if π_ϕ is selfadjoint then it is standard. We obtain that π_ϕ is always standard without the assumption of selfadjointness of π_ϕ. Further, we obtain that π_ϕ is selfadjoint if and only if π_ϕ is standard when π_{ϕ_α} is standard for all α.

2. Preliminaries. We state some definitions and notation used in this paper. Let ϕ be a positive linear functional on a *-algebra $\mathscr{A}$ and $(\pi_\phi, \lambda_\phi, \mathfrak{H}_\phi)$ the GNS-construction for ϕ. We now put

$$\mathscr{D}\left(\pi_\phi^*\right) = \bigcap_{x \in \mathscr{A}} \mathscr{D}\left(\pi_\phi(x)^*\right), \quad \pi_\phi^*(x)\xi = \pi_\phi(x^*)^*\xi \quad \text{for } x \in \mathscr{A}, \xi \in \mathscr{D}\left(\pi_\phi^*\right);$$

$$\mathscr{D}\left(\pi_\phi^{**}\right) = \bigcap_{x \in \mathscr{A}} \mathscr{D}\left(\pi_\phi^*(x)^*\right), \quad \pi_\phi^{**}(x)\xi = \pi_\phi^*(x^*)^*\xi \quad \text{for } x \in \mathscr{A}, \xi \in \mathscr{D}\left(\pi_\phi^{**}\right).$$

Then π_ϕ^* is a closed representation of $\mathscr{A}$, but it is not necessarily a *-representation; π_ϕ^{**} is a closed *-representation of $\mathscr{A}$; $\pi_\phi \subset \pi_\phi^{**} \subset \pi_\phi^*$, where $\pi_1 \subset \pi_2$ means that $\mathscr{D}(\pi_1) \subset \mathscr{D}(\pi_2)$ and $\pi_1(x)\xi = \pi_2(x)\xi$ for all $x \in \mathscr{A}$ and $\xi \in \mathscr{D}(\pi_1)$.

We define the notions of bounded, standard and selfadjoint representations as follows: π_ϕ is said to be bounded if $\pi_\phi(x)$ is a bounded operator on $\mathfrak{H}_\phi$ for all $x \in \mathscr{A}$; π_ϕ is said to be standard if $\pi_\phi(x)^* = \overline{\pi_\phi(x^*)}$ for all $x \in \mathscr{A}$; π_ϕ is said to be selfadjoint if $\pi_\phi = \pi_\phi^*$. It is clear that if π_ϕ is bounded then it is standard, and if π_ϕ is standard then it is selfadjoint.

3. Selfadjointness of the *-representation generated by the sum of two positive linear functionals. In this section let ϕ and ψ be positive linear functionals on a *-algebra $\mathscr{A}$ with identity e and $\tau = \phi + \psi$. We consider when π_τ is selfadjoint. The following theorem is an analogy of the well-known fact in the operator theory.

THEOREM 3.1. *Suppose that π_ϕ is selfadjoint and π_ψ is bounded. Then π_τ is selfadjoint.*

PROOF. We put

$$K_{\tau,\phi}\lambda_\tau(x) = \lambda_\phi(x) \quad \text{for } x \in \mathscr{A}.$$

Then $K_{\tau,\phi}$ is extended to the bounded linear map of $\mathfrak{H}_\tau$ into $\mathfrak{H}_\phi$, which is also denoted by $K_{\tau,\phi}$. Similarly we define the map $K_{\tau,\psi}$ of $\mathfrak{H}_\tau$ into $\mathfrak{H}_\psi$. Since $\tau = \phi + \psi$, we have

$$K_{\tau,\phi}^* K_{\tau,\phi} + K_{\tau,\psi}^* K_{\tau,\psi} = I, \tag{3.1}$$

where I denotes the identity operator on $\mathfrak{H}_\tau$. Since π_ϕ is selfadjoint and π_ψ is bounded, we can easily show that $\pi_\phi(x)K_{\tau,\phi} \supset K_{\tau,\phi}\pi_\tau^{**}(x)$ and $\pi_\psi(x)K_{\tau,\psi} \supset K_{\tau,\psi}\pi_\tau^{**}(x)$ for all $x \in \mathscr{A}$. Since π_ψ is bounded, it follows that $\pi_\psi(x)K_{\tau,\psi} = \overline{K_{\tau,\psi}\pi_\tau^{**}(x)} = K_{\tau,\psi}\pi_\tau(x^*)^*$ for all $x \in \mathscr{A}$. Hence we have

$$K_{\tau,\psi}\pi_\tau(x)^* \subset \pi_\psi(x)^* K_{\tau,\psi} \tag{3.2}$$

for all $x \in \mathscr{A}$, which implies that

$$K_{\tau,\psi}^* \xi \in \mathscr{D}\left(\overline{\pi_\tau(x)}\right) \quad \text{and} \quad \overline{\pi_\tau(x)}\, K_{\tau,\psi}^* \xi = K_{\tau,\psi}^* \pi_\psi(x) \xi \tag{3.3}$$

for each $x \in \mathscr{A}$ and $\xi \in \mathfrak{H}_\psi$. Then it follows from (3.1) that

$$\begin{aligned}
\left(\pi_\phi(x)\lambda_\phi(y) | K_{\tau,\phi}\eta\right) &+ \left(\pi_\phi(x)\lambda_\psi(y) | K_{\tau,\psi}\eta\right) \\
&= \left(K_{\tau,\phi}^* K_{\tau,\phi}\pi_\tau(x)\lambda_\tau(y) | \eta\right) + \left(K_{\tau,\psi}^* K_{\tau,\psi}\pi_\tau(x)\lambda_\tau(y) | \eta\right) \\
&= \left(\pi_\tau(x)\lambda_\tau(y) | \eta\right) \\
&= \left(\lambda_\tau(y) | \pi_\tau(x)^*\eta\right) \\
&= \left(\lambda_\phi(y) | K_{\tau,\phi}\pi_\tau(x)^*\eta\right) + \left(\lambda_\psi(y) | K_{\tau,\psi}\pi_\tau(x)^*\eta\right)
\end{aligned}$$

for each $\eta \in \mathscr{D}(\pi_\tau(x)^*)$ and $y \in \mathscr{A}$. By (3.3) we have

$$\left(\pi_\phi(x)\lambda_\phi(y) | K_{\tau,\phi}\eta\right) = \left(\lambda_\phi(y) | K_{\tau,\phi}\pi_\tau(x)^*\eta\right)$$

for all $x, y \in \mathscr{A}$ and $\eta \in \mathscr{D}(\pi_\tau(x)^*)$. Hence,

$$\pi_\phi(x)^* K_{\tau,\phi} \supset K_{\tau,\phi}\pi_\tau(x)^* \tag{3.4}$$

for all $x \in \mathscr{A}$, which implies that

$$K_{\tau,\phi}^* \eta \in \mathscr{D}\left(\overline{\pi_\tau(x)}\right) \quad \text{and} \quad \overline{\pi_\tau(x)}\, K_{\tau,\phi}^* \eta = K_{\tau,\phi}^* \pi_\phi(x) \eta \tag{3.5}$$

for each $x \in \mathscr{A}$ and $\eta \in \mathscr{D}(\pi_\phi)$. Take arbitrary $\eta \in \mathscr{D}(\pi_\tau^*)$. By (3.4) we have $K_{\tau,\phi}\eta \in \mathscr{D}(\pi_\phi)$, and hence it follows from (3.3) and (3.5) that $K_{\tau,\phi}^* K_{\tau,\phi}\eta \in \mathscr{D}(\pi_\tau)$ and $K_{\tau,\psi}^* K_{\tau,\psi}\eta \in \mathscr{D}(\pi_\tau)$, so that $\eta = K_{\tau,\phi}^* K_{\tau,\phi}\eta + K_{\tau,\psi}^* K_{\tau,\psi}\eta \in \mathscr{D}(\pi_\tau)$. Hence, π_τ is selfadjoint. This completes the proof.

As in the proof of Theorem 3.1 we can prove the following

THEOREM 3.2. *Suppose π_ϕ and π_ψ are standard. Then π_τ is selfadjoint if and only if π_τ is standard.*

By Theorems 3.1 and 3.2 we have the following

COROLLARY 3.3. *Suppose π_ϕ is standard and π_ψ is bounded. Then π_τ is standard.*

4. Standardness of *-representations generated by approximately admissible positive linear functionals. Let ϕ be a positive linear functional on a *-algebra $\mathscr{A}$ with identity e. When π_ϕ is bounded, ϕ is said to be admissible. If there exists a net $\{\phi_\alpha\}$ of admissible positive linear functionals on $\mathscr{A}$ such that $\phi_\alpha \leqq \phi$ for each α and $\lim_\alpha \phi_\alpha(x) = \phi(x)$ for each $x \in \mathscr{A}$, then ϕ is said to be approximately admissible.

Takesue proved in **[9]** that π_ϕ is selfadjoint if and only if π_ϕ is standard for every approximately admissible positive linear functional ϕ on $\mathscr{A}$. In this section we show that, for every approximately admissible positive linear functional ϕ on $\mathscr{A}$, π_ϕ is standard without the assumption of the selfadjointness of π_ϕ.

THEOREM 4.1. *Suppose that ϕ is an approximately admissible positive linear functional on $\mathscr{A}$. Then π_ϕ is standard.*

PROOF. Since ϕ is approximately admissible, there exists a net $\{\phi_\alpha\}$ of admissible positive linear functionals on $\mathscr{A}$ such that $\phi_\alpha \leqq \phi$ for each α and $\lim_\alpha \phi_\alpha(x) = \phi(x)$ for each $x \in \mathscr{A}$. Since π_{ϕ_α} is bounded for each α, it follows that

$$\pi_{\phi_\alpha}(x)^* K_{\phi,\phi_\alpha} \supset K_{\phi,\phi_\alpha}\pi_\phi(x)^* \tag{4.1}$$

for each α and $x \in \mathscr{A}$. Since $\phi_\alpha \leqq \phi$ for each α and $\lim_\alpha \phi_\alpha(x) = \phi(x)$ for each $x \in \mathscr{A}$, we see that $K^*_{\phi,\phi_\alpha}K_{\phi,\phi_\alpha}$ converges weakly to the identity operator I on $\mathfrak{H}_\phi$. Take arbitrary $x \in \mathscr{A}$. Since

$$\begin{aligned}\left(\pi_\phi(x)^*\xi | K^*_{\phi,\phi_\alpha}\eta\right) &= \left(K_{\phi,\phi_\alpha}\pi_\phi(x)^*\xi|\eta\right) \qquad \text{(by (4.1))}\\ &= \left(\pi_{\phi_\alpha}(x)^* K_{\phi,\phi b_\alpha}\xi|\eta\right)\\ &= \left(\xi | K^*_{\phi,\phi_\alpha}\pi_{\phi_\alpha}(x)\eta\right)\end{aligned}$$

for each $\xi \in \mathscr{D}(\pi_\phi(x)^*)$, $\eta \in \mathfrak{H}_{\phi_\alpha}$ and α, it follows that

$$K^*_{\phi,\phi_\alpha}\eta \in \mathscr{D}\left(\overline{\pi_\phi(x)}\right) \quad \text{and} \quad \overline{\pi_\phi(x)}\, K^*_{\phi,\phi_\alpha} \text{ then } \eta = K^*_{\phi,\phi_\alpha}\pi_{\phi_\alpha}(x)\eta \tag{4.2}$$

for each α and $\eta \in \mathfrak{H}_{\phi_\alpha}$, which implies that

$$\begin{aligned}\overline{\pi_\phi(x)}\, K^*_{\phi,\phi_\alpha}K_{\phi,\phi_\alpha}\xi &= K^*_{\phi,\phi_\alpha}\pi_{\phi_\alpha}(x)K_{\phi,\phi_\alpha}\xi \qquad \text{(by (4.1))}\\ &= K^*_{\phi,\phi_\alpha}K_{\phi,\phi_\alpha}\pi_\phi(x^*)^*\xi\end{aligned} \tag{4.3}$$

for each $\xi \in \mathscr{D}(\pi_\phi(x^*)^*)$ and α. Hence it follows that

$$\begin{aligned}\left(\pi_\phi(x)^*\zeta|\xi\right) &= \lim_\alpha\left(\pi_\phi(x)^*\zeta|K^*_{\phi,\phi_\alpha}K_{\phi,\phi_\alpha}\xi\right)\\ &= \lim_\alpha\left(\zeta|\,\overline{\pi_\phi(x)}\,K^*_{\phi,\phi_\alpha}K_{\phi,\phi_\alpha}\xi\right) \qquad \text{(by (4.2))}\\ &= \lim_\alpha\left(\zeta|K^*_{\phi,\phi_\alpha}K_{\phi,\phi_\alpha}\pi_\phi(x^*)^*\xi\right) \qquad \text{(by (4.3))}\\ &= \left(\zeta|\pi_\phi(x^*)^*\xi\right)\end{aligned}$$

for each $\zeta \in \mathscr{D}(\pi_\phi(x)^*)$ and $\xi \in \mathscr{D}(\pi_\phi(x^*)^*)$, which implies that $\overline{\pi_\phi(x)} = \pi_\phi(x^*)^*$ for each $x \in \mathscr{A}$. This completes the proof.

As in the proof of Theorem 4.1 we can prove the following

THEOREM 4.2. *Suppose ϕ is a positive linear functional on $\mathscr{A}$ such that there exists a net $\{\phi_\alpha\}$ of positive linear functionals on $\mathscr{A}$ satisfying: π_{ϕ_α} is standard for each α, $\phi_\alpha \leqq \phi$ for each α, and $\lim_\alpha \phi_\alpha(x) = \phi(x)$ for each $x \in \mathscr{A}$. Then π_ϕ is selfadjoint if and only if π_ϕ is standard.*

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