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A CANONICAL TRACE CLASS APPROXIMANT

D. A. LEGG AND J. D. WARD

ABSTRACT. Let H be a finite-dimensional Hilbert space, $B(H)$ the space of bounded linear operators on H, and C a convex subset of $B(H)$. If A is a fixed operator in $B(H)$, then A has a unique best approximant from C in the C_p norm for $1 < p < \infty$. However, in the C_1 (trace) norm, A may have many best approximants from C. In this paper, it is shown that the best C_p approximants to A converge to a select trace class approximant A_1 as $p \to 1$. Furthermore, A_1 is the unique trace class approximant minimizing $\sum_{i=1}^{n} S_i(A - B) \ln S_i(A - B)$ over all trace class approximants B. The numbers $S_i(T)$ are the eigenvalues of the positive part $|T|$ of T.

1. Introduction. The question of approximating a finite-dimensional matrix by operators from some prescribed convex set in the C_p norm, $1 < p < \infty$, has a simple answer. Namely, there is always a unique approximant due to the convexity of the class of approximating operators and the uniform rotundity of the C_p norm. However, in general a given matrix has a "large" set of approximants in the C_p norm for $p = 1$ or ∞.

The purpose of this paper is to show that the unique C_p approximants to a given finite-dimensional matrix A from a prescribed convex set converge to a "select" C_1 approximant as p tends to 1. This result answers, in the affirmative, questions 3 and 4 in [**8**]. In addition, the fact that the limit exists as $p \to 1$ establishes a canonical trace class approximant. Very often canonical approximants shed much light on the structure of the set of best approximants as seen in [**2**, **3** and **4**].

2. A canonical trace class approximant. In what follows, the term operator shall refer to a bounded linear operator on a complex Hilbert space H of dimension $n < \infty$. The numbers $S_1(A), \ldots, S_n(A)$ will denote the eigenvalues of $|A|$ listed in decreasing order, where A is a bounded linear operator on H having polar factorization $A = U|A|$. For $1 \leqslant p < \infty$, the C_p norm is defined by

$$\|A\|_p = \left(\sum_{i=1}^{n} S_i(A)^p \right)^{1/p}.$$

For a given matrix A, let A'_p denote a unique nearest point in the C_p norm from a closed convex set C and let $A_p = A - A'_p$. We establish that $\lim_{p \to 1} A_p$ (and hence $\lim_{p \to 1} A'_p$) exists in the trace norm.

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Motivated by the approach in [7] we let, for a fixed A,

$$\phi(A, p) = \|A\|_p^p = \sum_{i=1}^{n} S_i(A)^p.$$

Then

$$\frac{d\phi}{dp}(A, p) = \sum_{i=1}^{n} S_i(A)^p \ln S_i(A),$$

and so

$$\frac{d\phi}{dp}(A, 1) = \sum_{i=1}^{n} S_i(A) \ln S_i(A),$$

where ln denotes the natural log.

LEMMA 2.1. *If B is any cluster point of $\{A_p\}$ in the trace norm as $p \to 1$, then $A - B$ is a best C_1 approximant to A from C, and*

$$\sum_{i=1}^{n} S_i(B) \ln S_i(B) = \min_{D \in \mathscr{D}} \sum_{i=1}^{n} S_i(D) \ln S_i(D),$$

where $\mathscr{D} = \{A - D'\colon D'$ is a best C_1 approximant to A from $C\}$.

PROOF. Let $p_k \to 1$ with $A_{p_k} \to B$ in the trace norm. Clearly $A - B$ is a best C_1 approximant. Let $\Phi(x) = x \ln x$ for $x > 0$ with $\Phi(0) = 0$. Let $D_1 \in \mathscr{D}$ satisfy

$$\sum_{i=1}^{n} \Phi(S_i(D_1)) = \min_{D \in \mathscr{D}} \sum_{i=1}^{n} \Phi(S_i(D)).$$

It is easily checked that $\Phi(x) \leqslant (x^p - x)/(p - 1)$ for $x \geqslant 0$, $p > 1$, and so, for each $i = 1, \ldots, n$, we have

$$\Phi\big(S_i(A_{p_k})\big) \leqslant \frac{S_i(A_{p_k})^{p_k} - S_i(A_{p_k})}{p_k - 1}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{n} \Phi\big(S_i(A_{p_k})\big) &\leqslant \frac{1}{p_k - 1}\left[\sum_{i=1}^{n} S_i(A_{p_k})^{p_k} - \sum_{i=1}^{n} S_i(A_{p_k})\right] \\ &\leqslant \frac{1}{p_k - 1}\left[\sum_{i=1}^{n} S_i(D_1)^{p_k} - \sum_{i=1}^{n} S_i(D_1)\right] \\ &= \frac{1}{p_k - 1}\big[\phi(D_1, p_k) - \phi(D_1, 1)\big].\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\sum_{i=1}^{n} \Phi(S_i(B)) \leqslant \sum_{i=1}^{n} \Phi(S_i(D_1)).$$

Hence

$$\sum_{i=1}^{n} \Phi(S_i(B)) = \min_{D \in \mathscr{D}} \sum_{i=1}^{n} \Phi(S_1(D)).$$

THEOREM 2.2 *Let C be a closed convex set of $n \times n$ matrices and let A_p be the matrix in C with smallest C_p norm. Then* $\lim_{p\to 1} A_p$ *exists.*

PROOF. Since the underlying Hilbert space is finite dimensional, every sequence $\{A_{p_k}\}$, $p_k \to 1$, contains a convergent subsequence in the trace norm. Hence it suffices to show that $\{A_p\}$ has a unique cluster point as $p \to 1$.

Suppose B_1 and B_2 are cluster points of $\{A_p\}$ as $p \to 1$. If we multiply B_1 and B_2 by a sufficiently large positive scaling factor, we may assume that all the numbers

$$S_i(B_1/2), \qquad S_i(B_2/2), \qquad S_i(\tfrac{1}{2}(B_1 + B_2)),$$

$i = 1,\dots,n$, are either 0 or greater than 2.

Let $f(x)$ be a strictly increasing convex function such that $f(0) = 0$ and $f(x) = \Phi(x)$ for $x \geqslant 2$. By [5, Chapter 2, Theorem 4.1], we have

$$\begin{aligned}
\sum_{i=1}^{n} \Phi\left(S_i\left(\frac{1}{2}(B_1 + B_2)\right)\right) &= \sum_{i=1}^{n} f\left(S_i\left(\frac{1}{2}(B_1 + B_2)\right)\right) \\
&\leqslant \sum_{i=1}^{n} f\left(S_i\left(\frac{B_1}{2}\right) + S_i\left(\frac{B_2}{2}\right)\right) && (1) \\
&\leqslant \sum_{i=1}^{n} \left[\frac{1}{2} f(S_i(B_i)) + \frac{1}{2} f(S_i(B_2))\right] && (2) \\
&= \frac{1}{2}\sum_{i=1}^{n} \Phi(S_i(B_1)) + \frac{1}{2}\sum_{i=1}^{n} \Phi(S_i(B_2)).
\end{aligned}$$

Hence by Lemma 2.1, since B_1 and B_2 are the cluster points of $\{A_p\}$ as $p \to 1$, we have

$$\sum_{i=1}^{n} \Phi\left(S_i\left(\frac{1}{2}(B_1 + B_2)\right)\right) \leqslant \min_{D\in\mathscr{D}} \sum_{i=1}^{n} \Phi(S_i(D)).$$

Since $\frac{1}{2}(B_1 + B_2) \in \mathscr{D}$, it follows that

$$\sum_{i=1}^{n} \Phi\left(S_i\left(\frac{1}{2}(B_1 + B_2)\right)\right) = \min_{D\in\mathscr{D}} \sum_{i=1}^{n} \Phi(S_i(D)).$$

Hence both inequality signs in (1) and (2) must actually be equal signs.

The inequality (1) implies that

$$S_i(\tfrac{1}{2}(B_1 + B_2)) = S_i(B_1/2) + S_i(B_2/2)$$

for each $i = 1,\dots,n$ since f is strictly convex, while (2) implies that $S_i(B_1) = S_i(B_2) = S_i((B_1 + B_2)/2)$ for all i; i.e., $\|B_1\|_p = \|B_2\|_p = \|(B_1 + B_2)/2\|_p$. From the strict convexity of the C_p norm, it follows that $B_1 = B_2$. Hence there exists a unique cluster point of $\{A_p\}$ as $p \to 1$ and this completes the proof.

REMARK. A similar result seems likely in the case $p = \infty$ if the approximating class is the set of positive operators. We conjecture that the best C_p approximants converge in the operator norm to the approximant P_m as $p \to \infty$. See [8].

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY - PURDUE UNIVERSITY, FORT WAYNE, INDIANA 46805

DEPARTMENT OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TEXAS 77843