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ON THE COMBINATORIAL PROPERTIES OF BLACKWELL SPACES

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ABSTRACT. Under MA + $\neg$CH (Martin's Axiom and negation of the Continuum Hypothesis) we prove that the intersection of a Blackwell space with the analytic set and the Cartesian product of a Blackwell space and a Borel set do not need to be Blackwell spaces.

1. Introduction. In this paper we present a few examples illustrating the singular behaviour of Blackwell spaces. They shall provide (under MA + $\neg$CH) the negative answers to questions P4, P6, P7 and P8 raised by K. P. S. Bhaskara Rao and B. V. Rao in [**1**]. The first two were originally answered by W. Bzyl and J. Jasiński in [**4**]. Here we give a slight generalization of their result (Proposition 1).

On the other hand, W. Bzyl in [**3**], using the idea presented in [**9**], proved that questions P4, P6, P7 and a weaker version of P8 have positive answers when restricted to Blackwell spaces with totaly imperfect complement in some analytic set.

Let us recall the main definitions. For a metric space X by $\Sigma_\alpha X$, we denote the additive Baire classes of Borel subsets of X, and by $\mathscr{B}(X)$ we denote a σ-algebra of all Borel subsets of X. A Borel measurable mapping $f\colon X \to Y$, where Y is a metric space, is called a class Σ_α if, for all open subsets $U \subset Y$, $f^{-1}(U) \in \Sigma_\alpha(X)$.

A σ-algebra of subsets of X is called separable if it is countably generated (c.g.) and separates the points of X. Let X be a separable metric space. X is called a Blackwell space if $\mathscr{B}(X)$ does not contain a proper separable sub-σ-algebra. X is called a strongly Blackwell space if any two c.g. sub-σ-algebras of $\mathscr{B}(X)$ with the same atoms coincide. It is clear that a strongly Blackwell space is a Blackwell space. If A is an analytic subset of a Polish space, then A is strongly Blackwell. For this and for other results on Blackwell spaces see K. P. S. Bhaskara Rao and B. V. Rao [**1**].

2. Basic lemmas. We shall often refer to the well-known result of Silver:

LEMMA. 1. *(MA) If Z is a separable metric space and $|Z| < 2^\omega$, then $\mathscr{B}(Z) = \mathscr{P}(Z)$.*

For the proof see [**8**, pp. 162, 163]. As pointed out by K. P. S. Bhaskara Rao and B. V. Rao [**1**, p. 15], Lemma 1 implies the following

LEMMA 2. *(MA) If Z is a separable metric space with $|Z| < 2^\omega$, then Z is strongly Blackwell.*

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LEMMA 3. *If Y is a Blackwell space and B is a Borel subset of the Polish space, then $Y \cup B$ is a Blackwell space.*

For the proof see [**1**, p. 28, 2^0].

3. Main propositions. In this section, X will denote a Polish space.

PROPOSITION 1. *(MA) Let $B \in \mathscr{B}(X)$ be of cardinality 2^ω and let Z be an uncountable separable metric space of cardinality less than 2^ω. If $Z \cap B = \varnothing$, then $Z \cup B$ is a Blackwell space which is not strongly Blackwell.*

PROOF. By Lemma 2, Z is a Blackwell space, so, by Lemma 3, $B \cup Z$ is also a Blackwell space.

W. Bzyl and J. Jasiński in [**4**] proved that there exists a Borel set $B_1 \in \mathscr{B}(\mathbf{R}^2)$ and $Z_1 \subseteq \mathbf{R}$ with $|Z_1| = \omega_1$ such that $B_1 \cup Z_1$ is not a strongly Blackwell space. Let f: $B \to B_1$ be a Borel isomorphism (see [**6**, p. 450, Theorem 2]) and let

$$g\colon Z \underset{\text{onto}}{\to} Z_1.$$

By Lemma 1 a mapping h: $B \cup Z \to B_1 \cup Z_1$, defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \in B, \\ g(x) & \text{for } x \in Z, \end{cases}$$

is Borel measurable; hence, $B \cup Z$ is not strongly Blackwell.

A certain part of the next proposition does not require MA so we formulate it separately as

LEMMA 4. *Let $Y \subseteq X$ be a Blackwell (strongly Blackwell) space and let $Z \subseteq X \setminus Y$. If, for every Borel set $B \in \mathscr{B}(X)$, $B \cap Y = \varnothing$ implies $|B \cap Z| \leqslant \omega$, then $Y \cup Z$ is a Blackwell (strongly Blackwell) space.*[1]

PROOF. We give a proof in case Y is a strongly Blackwell space. Let $\mathscr{C} \subseteq \mathscr{D} \subseteq \mathscr{B}(Y \cup Z)$ be c.g. σ-algebras with the same atoms and let $D \in \mathscr{D}$. By [**1**, p. 23, Proposition 8(5)] it suffices to show that $D \in \mathscr{C}$. Since Y is strongly Blackwell, $\mathscr{C} \restriction_Y = \mathscr{D} \restriction_Y$, so there is a set $C \in \mathscr{C}$ such that

$$C \cap Y = D \cap Y. \tag{1}$$

Let C', $D' \in \mathscr{B}(X)$ be such that $C' \cap (Y \cup Z) = C$ and $D' \cap (Y \cup Z) = D$. By (1) the symmetric difference $D' \triangle C' \subseteq X \setminus Y$; hence, $|(D' \triangle C') \cap Z| \leqslant \omega$ and $|D \triangle C| \leqslant \omega$, so $D = C \triangle (C \triangle D) \in \mathscr{C}$.

Recall that whenever $A \subseteq X$ is an analytic non-Borel set, then there exist non-empty Borel sets C_α, $\alpha < \omega_1$, such that each Borel set $B \in \mathscr{B}(X)$ disjoint with A is covered by countably many C_α's. The sets C_α are called the constituents of a coanalytic set $X \setminus A$ (see [**6**, p. 499]).

Disjoint sets X_1, $X_2 \subseteq X$ are called Borel-separable if there is a Borel set $B \in \mathscr{B}(X)$ such that $X_1 \subseteq B$ and $X_2 \subseteq X \setminus B$.

[1] This lemma has been obtained independently by R. M. Shorttnnn and K. P. S. Bhaskara Rao.

PROPOSITION 2. (MA) *Let $A \subseteq X$ be an analytic non-Borel set and let $\{C_\alpha\}_{\alpha<\omega_1}$ be the constituents of $X \setminus A$. Whenever $Z \subseteq X$, $\omega < |Z| < 2^\omega$ and $A \cap Z = \varnothing$, then $A \cup Z$ is not a Blackwell space iff there is an $\alpha_0 < \omega_1$ such that $|C_{\alpha_0} \cap Z| > \omega$ and $|\{\alpha: C_\alpha \setminus Z \neq \varnothing\}| = \omega_1$.*

PROOF. "*If*" *part*. Let $Z_1 \subseteq Z \cap C_{\alpha_0}$ be of cardinality ω_1. Since

$$|\{\alpha < \omega_1: C_\alpha \setminus Z \neq \varnothing\}| = \omega_1,$$

there is a mapping $g\colon Z_1 \overset{1-1}{\to} X \setminus (A \cup Z)$ such that $g(Z_1)$ and A are disjoint non-Borel-separable.

Define $f\colon A \cup Z \overset{1-1}{\to} X$,

$$f(x) = \begin{cases} x & \text{for } x \in (A \cup Z) \setminus Z_1, \\ g(x) & \text{for } x \in Z_1. \end{cases}$$

By Lemma 1, f is Borel measurable, but $f(Z_1) = g(Z_1) \notin \mathscr{B}(f(A \cup Z))$, so $A \cup Z$ is not a Blackwell space (see [**1** p. 22, Proposition 7(2)]).

"*Only if*" *part*. Suppose $|\{\alpha: C_\alpha \setminus Z \neq \varnothing\}| < \omega_1$. In this case there is an $\alpha < \omega_1$ such that $Z \cup A = Z \cup (\bigcup_{\beta>\alpha} C_\beta) \cup A$ is Borel, so, by Lemmas 2 and 3, $A \cup Z$ is a Blackwell space.

In case, for every $\alpha < \omega_1$, $|C_\alpha \cap Z| < \omega_1$, then, for every Borel set $B \in \mathscr{B}(X)$, $B \cap A = \varnothing$ implies $|B \cap Z| \leqslant \omega$, so, by Lemma 4, $Z \cup A$ is a Blackwell space.

COROLLARY 1. (MA) *Let A and Z be as in Proposition 2. If A and Z are Borel separable, then $A \cup Z$ is not a Blackwell space.*

COROLLARY 2. $(MA + \neg\, CH)$ *There exists a Blackwell space Y and an analytic set $A \subset X$ such that $Y \cap A$ is not a Blackwell space.*

PROOF. Let $Y = B \cup Z$ where $B \in \mathscr{B}(X)$, $|B| = 2^\omega$, $Z \subset X$, $\omega < |Z| < 2^\omega$ and $B \cap Z = \varnothing$. By Lemmas 2 and 3, Y is a Blackwell space. Let $A_1 \subset B$ be an analytic non-Borel set and take $A = A_1 \cup (X \setminus B)$. $A \cap Y = A_1 \cup Z$ which is not Blackwell by Corollary 1.

PROPOSITION 3. (MA) *If $Z \subset X$ and $\omega < |Z| < 2^\omega$, then $Z \times B$ is not a Blackwell space, where B is an uncountable Borel subset of some Polish space.*

We shall precede the proof with two lemmas. The first one follows by Lemma 1 from [**2**, Theorem 3].

LEMMA 5. (MA) *Let $\mathscr{N}$ be a set of irrational numbers. If $Z \subseteq \mathscr{N}$ with $|Z| < 2^\omega$, then $B \in \mathscr{B}(Z \times \mathscr{N})$ iff there is an $\alpha < \omega_1$ such that for every $z \in Z$ the section $B_z = \{y: (z, y) \in B\} \in \Sigma_\alpha(\mathscr{N})$.*

Lemma 5 implies the following

LEMMA 6. (MA) *Let Z and B be as in Proposition 3. A mapping $h\colon Z \times B \to Y$, where Y is a separable metric space, is Borel measurable iff there is an $\alpha < \omega_1$ such that for every $z \in Z$ a restricted mapping $h \restriction \{z\} \times B$ is of class Σ_α.*

PROOF. Apply the isomorphism theorem [**6**, p. 450, Corollary 1c].

PROOF OF PROPOSITION 3. By the well-known theorem of Hausdorff [**5**], $\mathscr{N}$ can be decomposed into ω_1 disjoint uncountable sets of class $\Sigma_3(\mathscr{N})$, $\mathscr{N} = \bigcup_{\alpha < \omega_1} E_\alpha$.

Let $Z_1 \subset Z$ be of cardinality ω_1. By Lemma 1, $Z_1 \times B \in \mathscr{B}(Z \times B)$, so it suffices to prove that $Z_1 \times B$ is not a Blackwell space (see [**1**, p. 28, 1^0]). Let $Z_1 = \{z_\alpha : \alpha < \omega_1\}$. There is a $\gamma < \omega_1$ such that for each $\alpha < \omega_1$ there is a Borel measurable function f_α: $B \overset{1-1}{\to} E_\alpha$ of class Σ_γ (see [**6**, p. 450, Theorem 2]). By Lemma 6, a mapping h: $Z_1 \times \mathscr{N} \to \mathscr{N}$ defined by $h(z_\alpha, x) = f_\alpha(x)$ is Borel measurable, but the inverse mapping is not, since $Z_1 \times B$ is not Borel [**6**, p. 489, Theorem 1]. Hence, by [**1**, p. 22, Proposition 7 (2)], $Z_1 \times B$ is not a Blackwell space.

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