PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 4, April 1985

LINEAR MAPS DO NOT PRESERVE COUNTABLE-DIMENSIONALITY

MLADEN BESTVINA AND JERZY MOGILSKI

ABSTRACT. Examples of linear maps between normed spaces are constructed, including a one-to-one map from a countable-dimensional linear subspace of l_2 onto l_2. We prove that the linear span of a countable-dimensional linearly independent subset of a normed linear space is, in many cases, countable dimensional.

1. Introduction. In this note we shall prove that for a given separable Banach space Y there exists a one-to-one, continuous linear surjection $F\colon E \to Y$, where E is a normed linear space, which is a countable union of zero-dimensional sets. The space E will be obtained as the linear span of a carefully embedded zero-dimensional metric space into a Banach space. If Y is a Hilbert space then E can be chosen to be a linear subspace of the Hilbert space.

In the proof we use the well-known construction of an embedding of a metric space onto a linearly independent subset of a Banach or Hilbert space briefly described in §2. In §3 we will prove that the linear span of a carefully embedded countable-dimensional, separable metric space is also countable dimensional. In §4 we will construct some examples of linear maps "raising" topological dimension.

2. The standard embedding into l_p-spaces, $1 \leqslant p \leqslant \infty$. Recall that for a set S we can define normed spaces $l_p(S)$, $1 \leqslant p \leqslant \infty$. For $1 \leqslant p < \infty$, $l_p(S)$ consists of functions $z\colon S \to \mathbf{R}$ such that $\Sigma_{s \in S}|z(s)|^p < \infty$ with usual addition and scalar multiplication. The p-norm of $z \in l_p(S)$ is $\|z\|_p = (\Sigma_{s \in S}|z(s)|^p)^{1/p}$.

The space $l_\infty(S)$ consists of all bounded functions $z\colon S \to \mathbf{R}$. The ∞-norm of $z \in l_\infty(S)$ is $\|z\|_\infty = \sup_{s \in S}|z(s)|$.

Let X be a metric space with metric d bounded by 1. In this section we briefly describe an embedding $h\colon X \to l_p(S)$, $1 \leqslant p \leqslant \infty$, with certain nice properties.

First consider the case $1 \leqslant p < \infty$. The construction for $p = 2$ can be found in [**BP**, p. 193].

For $n = 1, 2, \ldots$ fix a locally finite partition of unity $\{\phi\}_{\phi \in \Lambda_n}$ such that $d(x, y) \geqslant 1/2^n$ implies $\phi(x) \cdot \phi(y) = 0$ for all $\phi \in \Lambda_n$. Then for $x \in X$ define $\hat{x}\colon \Lambda \to \mathbf{R}$,

Received by the editors April 10, 1984. Partially presented at a Special Session of the 810th meeting of the AMS at the University of Notre Dame, April 6–7, 1984.

1980 *Mathematics Subject Classification*. Primary 46B99, 54F45; Secondary 54C25, 55P10.

Key words and phrases. Countable dimension, normed linear spaces, continuous linear maps, Hamel basis, linear span, embeddings onto linearly independent sets.

where $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$, by $\hat{x}(\phi) = [1/2^n \cdot \phi(x)]^{1/p}$, for $\phi \in \Lambda_n$. Then

$$\hat{x} \in l_p(\Lambda) \quad \text{and} \quad \|\hat{x}\|_p = \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \left(\sum_{\phi \in \Lambda_n} \phi(x) \right) \right)^{1/p} = 1.$$

2.1. PROPOSITION. *The function h: $X \to l_p(\Lambda)$ given by $h(x) = \hat{x}$ has the following properties.*

(1) *h is an embedding,*

(2) *$h(X)$ is a linearly independent subset, and*

(3) *for every $x \in X$, and every closed subset $F \subseteq X$ with $x \notin F$, there exists a continuous linear functional ψ: $l_p(\Lambda) \to \mathbf{R}$ such that $\psi(h(F)) = \{0\}$ and $\psi(h(x)) \neq 0$.*

For the proof of (1) and (2) (for the case $p = 2$), see [**BP**, p. 193]. Property (3) is built into the construction, since we can use the "projection onto a ϕ-coordinate", i.e. the functional $\psi(z) = z(\phi)$ for appropriately chosen $\phi \in \Lambda_n$. (Pick $n \geqslant 1$ such that $1/2^n < d(x, F)$, and find $\phi \in \Lambda_n$ such that $\phi(x) \neq 0$.)

Note that if X is a separable metric space, we can arrange that Λ is countable.

The construction for $p = \infty$ can also be found in [**BP**, p. 49]. Define the space $Y = X \cup \{y_0\}$, with the metric $\tilde{d}$ that extends d and has the property that $\tilde{d}(x, y_0) = 1$ for $x \in X$. Let $A = \{\alpha\colon Y \to \mathbf{R};\ \alpha(y_0) = 0, |\alpha(y_1) - \alpha(y_2)| \leqslant \tilde{d}(y_1, y_2)$ for all $y_1, y_2 \in Y\}$. Finally, define h: $X \to l_\infty(A)$ by $h(x) = \hat{x}$, where $\hat{x}(\alpha) = \alpha(x)$.

2.2 PROPOSITION. *The map h is an isometry, $h(X)$ is a linearly independent subset of $l_\infty(A)$, and*

(4) *for every $x \in X$ and every closed subset $F \subseteq X$ with $x \notin F$ there exists a continuous linear functional ψ: $l_\infty(A) \to \mathbf{R}$ such that $\psi(h(F)) = \{0\}$ and $\psi(h(x)) \neq 0$.*

Again, (4) can be proved using the appropriate projection. If we set $\alpha(y) = \tilde{d}(y, F \cup \{y_0\})$, then the functional ψ: $l_\infty(A) \to \mathbf{R}$ defined by $\psi(z) = z(\alpha)$, has the desired property. The rest is proved in [**BP**].

3. Countable-dimensional linear spaces. We can construct many interesting normed spaces by taking span $h(X)$ where h: $X \to E$ is an embedding of a metric space into a normed space such that $h(X)$ is a linearly independent subset (e.g. we can use the construction described in §2). The question we want to address in this section is: When is span $h(X)$ countable dimensional? (A separable metric space Z is countable dimensional if it can be represented as a countable union of zero-dimensional subsets.) The obvious necessary condition is that X must be countable dimensional.

3.1. EXAMPLE. Choose a Hamel basis X of $l_2 = l_2(\mathbf{N})$, and let f: $C \to X$ be a one-to-one surjective map from a zero-dimensional separable metric space C. Assuming that $C \subseteq [\frac{1}{2}, 1]$, we set

$$X' = \left\{ \frac{f^{-1}(x)}{\|x\|_2} \cdot x\colon x \in X \right\}.$$

Then X' is also a Hamel basis for l_2, and $x' \mapsto \|x'\|_2$ defines a homeomorphism $X' \approx C$. Therefore dim $X' = 0$ and span $X' = l_2$ (which is not countable dimensional).

It is known (cf. [**BP**, p. 282]) that if X is a countable union of finite-dimensional compacta, then span $h(X)$ is countable dimensional (for every embedding $h: X \to E$ such that $h(X)$ is a linearly independent subset of E). We prove in this section that if h is a "nice" embedding, then span $h(X)$ is countable dimensional, provided X is countable dimensional. The standard embeddings described in §2 possess this nice property.

3.2. THEOREM. *Let h: $X \to E$ be an embedding of a countable dimensional separable metric space X into a linear metric space E such that $h(X)$ is a linearly independent subset of E. Suppose that $h(X)$ satisfies the following property.*

$(*)$ *For every $x \in X$ and every closed subset $F \subseteq X$ with $x \notin F$ there exists a continuous linear functional ψ: $E \to \mathbf{R}$ such that $\psi(h(F)) = \{0\}$ but $\psi(h(x)) \neq 0$.*

Then span $h(X) \subseteq E$ *is countable dimensional.*

PROOF. To an ordered collection $(N; i_1,\dots,i_s)$ of positive integers with $i_1 < \cdots < i_s$ we assign the collection $T(N; i_1,\dots,i_s) = \{(t_1,\dots,t_m) \in \mathbf{R}^m\colon -N \leqslant t_1 = \cdots = t_{i_1}, t_{i_1} + 1/N \leqslant t_{i_1+1} = \cdots = t_{i_2}, t_{i_2} + 1/N \leqslant t_{i_2+1} = \cdots = t_{i_3},\dots,t_{i_{s-1}} + 1/N \leqslant t_{i_{s-1}+1} = \cdots = t_{i_s} = t_m \leqslant N, |t_i| \geqslant 1/N \text{ for all } i\}$. Denote by $X(N; i_1,\dots,i_s)$ the collection of points z in span $h(X)$ that can be represented as $z = t_1h(x_1) + \cdots + t_mh(x_m)$ for some $(t_1,\dots,t_m) \in T(N; i_1,\dots,i_s)$, and some $(x_1,\dots,x_m) \in X^m$ with $x_i \neq x_j$ for $i \neq j$. Note that span $h(X) - \{0\}$ can be represented as the countable union of such subsets (for different choices of $(N; i_1,\dots,i_s)$). Consequently, it suffices to prove that $X(N; i_1,\dots,i_s)$ is countable dimensional.

Define a map χ: $\{(x_1,\dots,x_m) \in X^m\colon x_i \neq x_j \text{ for } i \neq j\} \times T(N; i_1,\dots,i_s) \to X(N; i_1,\dots,i_s)$ by

$$\chi(x_1,\dots,x_m, t_1,\dots,t_m) = t_1h(x_1) + \cdots + t_mh(x_m).$$

Noting that the domain of χ is countable dimensional (since it is contained in $[-N, N]^m \times X^m$), the rest of the proof follows from the next two lemmas.

3.3. LEMMA. *χ is a closed $i_1!(i_2 - i_1)! \cdots (i_s - i_{s-1})!$-to-1 surjection.*

3.4. LEMMA. *If f: $X \to Y$ is a closed q-to-1 map between separable metric spaces $(q \geqslant 1)$, and if X is countable dimensional, then Y is countable dimensional.*

PROOF OF LEMMA 3.3. From the uniqueness of the representation of $z \in$ span $h(X) - \{0\}$ as a linear combination of elements in $h(X)$ (up to a permutation), it follows that χ is a $i_1!(i_2 - i_1)! \cdots (i_s - i_{s-1})!$-to-1 surjection. To show that χ is closed, it suffices to prove that if $(z_k)_{k=1}^{\infty}$ is a sequence in the domain of χ, and if $\chi(z_k) \to \chi(z)$ for some z in the domain of χ, then $(z_k)_{k=1}^{\infty}$ has a convergent subsequence. To set the notation, let $z_k = (x_1^k,\dots,x_m^k, t_1^k,\dots,t_m^k)$, $z = (x_1,\dots,x_m, t_1,\dots,t_m)$. Passing to a subsequence, we may assume that

(5) $t_i^k \to t_i^0$, $i = 1,\dots,m$, and

(6) $(x_i^k)_{k=1}^{\infty}$ either converges, or does not have a convergent subsequence, $i = 1,\dots,m$.

For $i = 1,\dots,m$ let $\Omega_i = \{j\colon x_j^k \to x_i\}$. By $(*)$ we can choose a linear functional $\psi\colon E \to \mathbf{R}$ such that

(7) $\psi(h(x_i)) \neq 0$,

(8) $\psi(h(x_j)) = 0$, for $j \neq i$, and

(9) $\psi(h(x_j^k)) = 0$, for all $j \notin \Omega_i$ and all but finitely many values of k.

Passing to the limit of the left-hand side in $\psi\chi(z_k) \to \psi\chi(z)$ it follows that

$$\sum_{j\in\Omega_i} t_j^0 = t_i. \tag{10}$$

Since $t_i \neq 0$, we must have $\Omega_i \neq \phi$ $(i = 1,\dots,m)$. Moreover, since $\Omega_1,\dots,\Omega_m$ are pairwise disjoint subsets of $\{1,\dots,m\}$, it follows that card $\Omega_i = 1$, $i = 1,\dots,m$, and hence $\Omega_1 \cup \cdots \cup \Omega_m = \{1,\dots,m\}$. In particular, $(z_k)_{k=1}^{\infty}$ converges.

Proof of Lemma 3.4. For $p = 1,2,\dots$ denote $Y_p = \{y \in Y\colon d(x,x') \geqslant 1/p$ for all $x, x' \in f^{-1}(y)$ with $x \neq x'\}$. Since $Y = Y_1 \cup Y_2 \cup \cdots$ it suffices to show that Y_p is countable dimensional for each p. We will prove that $f|f^{-1}(Y_p)\colon f^{-1}(Y_p) \to Y_p$ is a local homeomorphism (and hence, by separability, Y_p can be covered by countably many open sets, each of which embeds into $f^{-1}(Y_p) \subseteq X$). For $x \in f^{-1}(Y_p)$ consider

$$f|\colon \overline{N_{1/3p}(x)} \cup f^{-1}(Y_p) \to f\left(\overline{N_{1/3p}(x)}\right) \cap Y_p$$

$\left(\overline{N_{1/3p}(x)}\right.$ is the closed $1/3p$-ball about x). Clearly, this a closed one-to-one surjection. To finish the argument, observe that $f\left(\overline{N_{1/3p}(x)}\right) \cap Y_p$ contains a neighborhood of $f(x)$ in Y_p. (If $y \in Y_p$ is close enough to $f(x)$, then $f^{-1}(y)$ is contained in $1/3p$-neighborhood of $f^{-1}f(x)$. Since $y \in Y_p$, no pair of points of $f^{-1}(y)$ can be contained in $1/3p$-neighborhood of some $x' \in f^{-1}f(x)$. Using that card $f^{-1}(y) =$ card $f^{-1}f(x)$, it follows that $f^{-1}(y)$ intersects $\overline{N_{1/3p}(x)}$.)

4. Examples of linear maps raising topological dimension. We will need the following observation concerning linear extension of continuous maps.

4.1 Lemma. *Let X be a Hamel basis in a normed linear space $(E, |\ |_1)$ and let $F\colon E \to Y$ be a linear map of E into a Banach space $(Y, |\ |_2)$ given by*

$$F\left(\sum_{i=1}^{n} t_i x_i\right) = \sum_{i=1}^{n} t_i f(x_i), \quad \text{where } x_i \in X,\ t_i \text{ is a real number for } i = 1,\dots,n,$$
$$\text{and } f\colon X \to Y \text{ is a continuous map}.$$

Let $|\ |$ be a new norm on E defined by $|x| = (|x|_1^2 + |F(x)|_2^2)^{1/2}$. Then:

(i) *the map $F\colon (E, |\ |) \to (Y, |\ |_2)$ is continuous,*

(ii) *if the Hamel basis X satisfies the condition $(*)$ of* 3.2 *with respect to the norm $|\ |_1$, then X satisfies $(*)$ with respect to $|\ |$,*

(iii) *the norm $|\ |$ induces the same topology on X.*

Let us observe that F may not be continuous as a map of $(E, |\ |_1)$ into $(Y, |\ |_2)$. For instance, let $E = \operatorname{span} X$, where $X = \{(x_i) \in l_2\colon x_i = t^i,\ t \in [\frac{1}{3}, \frac{2}{3}]\}$. Let f be a continuous real-valued function on X such that $f^{-1}(0) = \{(t^i) \in X\colon t \in [\frac{1}{3}, \frac{1}{2}]\}$ and $f^{-1}(1) = ((\frac{2}{3})^i)$. The set X is a Hamel basis for E and span $f^{-1}(0)$ is dense in E (cf. [**BP**, p. 267]). Hence the linear extension F of f is not continuous because $F(x) = 0$ for $x \in \operatorname{span} f^{-1}(0)$ and $F(((\frac{2}{3})^i)) = 1$.

Let us recall that a linear subspace of the Hilbert space is called a pre-Hilbert space.

4.2. EXAMPLE. There exists a continuous one-to-one linear surjection $F: E \to l_2$ of a countable-dimensional pre-Hilbert space E onto l_2.

PROOF. Let X be the zero-dimensional Hamel basis of the Hilbert space l_2 constructed in §3, and let $h: X \to l_2$ be an embedding of X onto linearly independent subset of l_2 described in §2. Let us consider the linear space $E = \operatorname{span} h(X)$ with the norm given by $|y| = (\|y\|_2^2 + \|F(y)\|_2^2)^{1/2}$ for $y \in E$, where $F: E \to l_2$ is the linear extension of the map $h^{-1}: h(X) \to X$. (Note that the norm $|\ |$ is induced by an inner product $x * y = x \cdot y + F(x) \cdot F(y)$ for $x, y \in E$. Thus the linear completion of E is isomorphic to l_2, and hence E is a pre-Hilbert space.)

By Lemma 4.1 and Theorem 3.2, $(E, |\ |)$ is a countable-dimensional pre-Hilbert space, and the linear map $F: (E, |\ |) \to (l_2, \|\ \|_2)$ is a continuous, one-to-one surjection.

Repeating the above construction we obtain

4.3. EXAMPLE. Let Y be a separable Banach space. There exists a continuous, one-to-one linear surjection $F: E \to Y$ of a countable-dimensional normed linear space E onto Y.

A metric space X is σ-finite-dimensional-compact if X is a countable union of finite-dimensional compacta. The next example answers a question posed in [**MM**].

4.4 EXAMPLE. There exists a continuous linear surjection F of a σ-finite-dimensional-compact pre-Hilbert space V onto the pre-Hilbert space $\Sigma = \{(t_i) \in l_2: \sum_{i=1}^{\infty}(it_i)^2 < \infty\}$ which contains the infinite-dimensional compact convex set $Q = \{(t_i) \in l_2: \sum_{i=1}^{\infty}(it_i)^2 \leqslant 1\}$.

PROOF. Let $f: I \to Q$ be a continuous surjection of the interval $I = [0,1]$ onto Q and let $h: I \to l_2$ be an embedding onto a linearly independent subset of l_2 described in §2. The linear extension $F: \operatorname{span} h(I) \to \Sigma$ of the map $fh^{-1}: h(I) \to Q$ is a continuous linear surjection of the σ-finite-dimensional-compact pre-Hilbert space $V = (\operatorname{span} h(I), |\ |)$ onto Σ, where $|x| = (\|x\|_2^2 + \|F(x)\|_2^2)^{1/2}$ (see [**BP**, p. 282] for the proof that span $h(I)$ is σ-finite-dimensional-compact).

4.5. EXAMPLE (CF. [**MM**]). There exists an open linear surjection of a σ-finite-dimensional-compact pre-Hilbert space onto a pre-Hilbert space which is not countable dimensional.

PROOF. Let $F: V \to \Sigma$ be the map constructed in Example 4.4. Let $Y = \sigma/\operatorname{Ker} F$ be the quotient space. Then the quotient map $T: V \to Y$ is open. The space Y cannot be countable dimensional because it is σ-compact and we can map Y onto Σ by a continuous, one-to-one map.

4.6. REMARK. Each continuous linear map between metric linear spaces is a UV^{∞}-map (a map $f: X \to Y$ is a UV^{∞}-map if for every $y \in Y$ and every open set U containing y, there exists an open set V, $y \in V \subset U$, such that $f^{-1}(V)$ is contractible in $f^{-1}(U)$). By [**H**] the linear maps constructed in §4 are fine homotopy equivalences (the map $f: X \to Y$ is a fine homotopy equivalence if for every open cover U of Y there exists a map $g: Y \to X$ such that $f \circ g$ is U-homotopic to id_Y and $g \circ f$ is $f^{-1}(U)$-homotopic to id_X). Hence the Examples 4.2, 4.3 show that even one-to-one fine homotopy equivalences can raise dimension (cf. [**A**]).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916

INSTITUTE OF MATHEMATICS, UNIVERSITY PKiN, 00 - 901 WARSAW, POLAND (Current address of Jerzy Mogilski)

Current address (Mladen Bestvina): Department of Mathematics, University of California, Berkeley, California 94720