PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 93, Number 4, April 1985

FIBRE TENSOR PRODUCT BUNDLES

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ABSTRACT. In analogy with fibre bundles, which are locally Cartesian products, fibre tensor product bundles are objects that are locally tensor products. These can be patched together *via* transition maps, etc., into an object very similar to the set of sections of a locally convex algebra bundle.

0. Introduction. It is known [**4**, **5**] that for large classes of topological algebras the following kind of relationships obtain: $\mathrm{Hom}(A \otimes_C B, \mathbf{C})$ is the pull-back of the diagram:

$$\begin{array}{ccc} & & \mathrm{Hom}(A,\mathbf{C}) \\ & & \downarrow \alpha \\ \mathrm{Hom}(B,\mathbf{C}) & \underset{\beta}{\rightarrow} & \mathrm{Hom}(C,\mathbf{C}) \end{array}$$

When $C = \mathbf{C}$, the pull-back is simply $\mathrm{Hom}(A,\mathbf{C}) \times \mathrm{Hom}(B,\mathbf{C})$. Furthermore [**5**, **7**, **10**] if $\mathscr{E}$ is a Banach algebra bundle (base space X compact Hausdorff, projection p: $\mathscr{E} \to X$, fibre A a commutative or Q-uniform Banach algebra, open cover $\{U_\lambda\}$ of X, transition maps $h_{\lambda\mu}$: $U_\lambda \cap U_\mu \to \mathrm{Aut}(A)$), then the set Γ of sections γ: $X \to \mathscr{E}$ is a Banach algebra and $\mathrm{Hom}(\Gamma,\mathbf{C})$ is also a fibre bundle (base space X, projection π: $\mathrm{Hom}(\Gamma,\mathbf{C}) \to X$, fibre $\mathrm{Hom}(A,\mathbf{C})$, open cover $\{U_\lambda\}$ of X, transition maps $H_{\lambda\mu}$: $U_\lambda \cap U_\mu \to \mathrm{Auteo}(\mathrm{Hom}(A,\mathbf{C})))$. For each U_λ, $\Gamma|_{U_\lambda}$ is a subalgebra of $\mathscr{C}(U_\lambda, A)$ which, in turn, is closely related to some kind of tensor product $\mathscr{C}(U_\lambda) \otimes A$. Thus, one is led to regard Γ as having a "fibre bundle-like" structure, that is locally like a tensor product. In turn, one is led thus to the notion of a fibre tensor product bundle and the study of such an object in the context of topological algebra.

1. Fibre tensor product bundles. Let A and B be two unital locally convex algebras such that $\mathrm{Hom}(A,\mathbf{C})$ = spectrum of A is compact. Associated with A and B are the following:

(i) *In B, a finite set $\{I_\lambda\}$ of closed 2-sided ideals, having compact hulls $\{h(I_\lambda) = F_\lambda\} \subset \mathrm{Hom}(B,\mathbf{C})$, the interiors $\{F_\lambda^\circ = U_\lambda\}$ are nonempty and $\bigcup_\lambda U_\lambda = \mathrm{Hom}(B,\mathbf{C})$.*

(ii) *Continuous transition maps $\bar{h}_{\lambda\mu}$: $U_\lambda \cap U_\mu \to \mathrm{Aut}(A)$ = the group of continuous $\mathbf{C}$-automorphisms of A; on $U_\lambda \cap U_\mu \cap U_\nu$, $\bar{h}_{\lambda\mu} \circ \bar{h}_{\mu\nu} = \bar{h}_{\lambda\nu}$; each $\bar{h}_{\lambda\mu}$ has a continuous extension $h_{\lambda\mu}$: $F_\lambda \cap F_\mu \to \mathrm{Aut}(A)$ behaving on $\{F_\lambda\}$ as the $\bar{h}_{\lambda\mu}$ behave on $\{U_\lambda\}$.*

Received by the editors December 23, 1983.
1980 *Mathematics Subject Classification*. Primary 46M05.

(iii) *The locally convex algebra direct sum* S, *namely if* $B_\lambda = B/I_\lambda$, $S = \sum_\lambda B_\lambda \otimes_\pi A$, $\otimes_\pi$ *denoting the projective tensor product* [**8**]; *in* S *a closed subalgebra* D *consisting of all* $\tau = (\tau_\lambda)$ *in* S *and such that if* $F_\lambda \cap F_\mu \neq \varnothing$, *then for some* f *in* $F_\lambda \cap F_\mu$ *and* $\tau_\lambda = \sum_n b_{\lambda n} \otimes a_{\lambda n}$,

$$\sum_n \hat{b}_{\lambda n}(f) a_{\lambda n} = \sum_n \hat{b}_{\mu n}(f) h_{\lambda\mu} a_{\mu n}.$$

The situations (i)–(iii) describe a coordinate fibre tensor product bundle $\mathscr{B}$. The object D, being the counterpart of the set of sections of an algebra bundle, is "locally" like $B_\lambda \otimes_\pi A$, and its localities are matched via the automorphisms $h_{\lambda\mu}$ in the manner described above.

The situation described above is exemplified if A and B are unital locally convex Q-algebras (locally convex algebras for which the group of units (invertible elements) is an open set [**8**]).

It will be shown that $\mathrm{Hom}(D, \mathbf{C})$ is (equivalent to) a kind of fibre bundle (base space $\mathrm{Hom}(B, \mathbf{C})$, projection p: $\mathrm{Hom}(D, \mathbf{C}) \to \mathrm{Hom}(B, \mathbf{C})$, fibre $\mathrm{Hom}(A, \mathbf{C})$, open cover $\{U_\lambda\}$, transition maps $h^*_{\mu\lambda}$: $U_\lambda \cap U_\mu \to \mathrm{Auteo}(\mathrm{Hom}(A, \mathbf{C}))$ = the group of self-homeomorphisms of $\mathrm{Hom}(A, \mathbf{C})$), equipped with a suitable topology, making it a toplogical transformation group and $h^*_{\mu\lambda}$ continuous (cf. comments after Lemma 1 in §2 below). Thus, the fibre tensor product is akin to the abstract algebraic version of the set Γ of continuous sections of a Banach (or, more generally, locally convex [7]) algebra bundle.

2. The theorems and their proofs.

LEMMA 1. *If* T *is a* $\mathbf{C}$*-automorphism of a locally convex algebra* A, *then* T^*: $\mathrm{Hom}(A, \mathbf{C}) \ni \chi \mapsto T^*(\chi) \in \mathrm{Hom}(A, \mathbf{C})$ $(T^*(\chi)a = \chi(T(a)))$ *is an auteomorphism of* $\mathrm{Hom}(A, \mathbf{C})$.

PROOF. The bijectivity of T^* follows from that of T and the bicontinuity of T^* follows from the definition of the weak topology on $\mathrm{Hom}(A, \mathbf{C})$. ■

Denote by $\mathscr{A}^*$ the *relevant* part of $\mathscr{A} \equiv \mathrm{Auteo}(\mathrm{Hom}(A, \mathbf{C}))$ defined by the above lemma, i.e. $\mathscr{A}^* = \{T^*\colon T \in \mathrm{Aut}(A)\}$. Then since $\mathrm{Aut}(A)$ and $\mathscr{A}^*$ are bijectively related via $T \mapsto T^*$, $\mathscr{A}^*$ may be given the topology it inherits from $\mathrm{Aut}(A)$ by this bijection, and then the continuity of $h_{\lambda\mu}$: $F_\lambda \cap F_\mu \to \mathrm{Aut}(A)$ implies the continuity of the (dual) maps $h^*_{\mu\lambda}$: $F_\lambda \cap F_\mu \to \mathscr{A}^*$. Now, it is assumed that $\mathrm{Aut}(A)$ is an equicontinuous subset of $\mathscr{L}_s(A)$ (the set of linear endomorphisms of A; $\mathscr{L}_s(A)$ is topologized by convergence on finite sets). This assumption is valid, e.g., in case of a Fréchet locally convex algebra, when by $\mathrm{Aut}(A)$ one should mean, of course, auteomorphisms of the whole structure, namely isometries [**2**, Example 2, p. 22]. Hence $\mathrm{Aut}(A)$ is a topological transformation group. Furthermore, if $\mathscr{A}$ is equipped with the topology of convergence on compact sets, then $\mathscr{A}$ is a topological transformation group and the (dual) maps $h^*_{\mu\lambda}$: $F_\lambda \cap F_\mu \to \mathscr{A}$ are also continuous. Indeed, by [**10**, §5.4, p. 19] $\mathscr{A}$ is a topological transformation group and, moreover, $\mathscr{A}^*$ is also a topological transformation group since equicontinuity implies that the topologies of convergence on finite and on compact sets are the same [**2**, §2.4, Theorem 1, p. 29]. Thus, the transition maps $h^*_{\mu\lambda}$: $F_\lambda \cap F_\mu \to \mathrm{Auteo}(\mathrm{Hom}(A, \mathbf{C}))$ are, in fact, continuous.

LEMMA 2. *If I_λ, I_μ are closed ideals in the unital locally convex algebra B, then there is an injection*

$$\operatorname{Hom}(B/I_\lambda \otimes_\pi B/I_\mu, \mathbf{C}) \hookrightarrow \operatorname{hull}(I_\lambda + I_\mu).$$

PROOF. Since B is unital (with identity 1_B) the map $I_\lambda \ni x_\lambda \mapsto x_\lambda \otimes 1_B$ is an injection $I_\lambda \hookrightarrow I_\lambda \otimes B$ and in these circumstances ($I_\lambda + I_\mu$ denoting the closed subspace spanned by I_λ and I_μ) there is an injection $I_\lambda + I_\mu \hookrightarrow I_\lambda \otimes B + B \otimes I_\mu$, whence there is a continuous epimorphism

$$\psi\colon B/I_\lambda + I_\mu \to B/I_\lambda \otimes_\pi B + B \otimes_\pi I_\mu.$$

On the other hand, since B is unital there are bicontinuous isomorphisms

$$B/I_\lambda \otimes_\pi B + B \otimes_\pi I_\mu \underset{\omega_2}{\to} B \otimes_\pi B/I_\lambda \otimes_\pi B + B \otimes_\pi I_\mu \underset{\omega_1}{\to} B/I_\lambda \otimes_\pi B/I_\mu,$$

as the following lines show.

In the following diagram π_2, π_2', π_λ, π_μ are the canonical quotient maps and φ, φ' are the canonical maps of the corresponding tensor products:

$$\begin{tikzcd}
B/I_\lambda + I_\mu \arrow[r, "\psi"] & B/I_\lambda \otimes_\pi B + B \otimes_\pi I_\mu \arrow[r, "\omega_2"] & B \otimes_\pi B/I_\lambda \otimes_\pi B + B \otimes_\pi I_\mu \arrow[r, "\omega_1"] & B/I_\lambda \otimes_\pi B/I_\mu \\
B \arrow[u, "\pi_2"] \arrow[r, "\omega_2'"'] & B \otimes_\pi B \arrow[ur, "\pi_2'"] \arrow[urr, "\pi_\lambda \otimes \pi_\mu"'] & B \times B \arrow[l, "\varphi"] \arrow[r, "\pi_\lambda \times \pi_\mu"'] & B/I_\lambda \times B/I_\mu \arrow[u, "\varphi'"']
\end{tikzcd}$$

DIAGRAM 1

Then ω_1 is continuous iff $\omega_1 \circ \pi_2'$ is continuous and since $\omega_1 \circ \pi_2' = \pi_\lambda \otimes \pi_\mu$, ω_1 is continuous; ω_1^{-1} is continuous iff $\omega_1^{-1} \circ \varphi'$ is continuous iff $(\omega_1^{-1} \circ \varphi') \circ (\pi_\lambda \times \pi_\mu)$ is continuous; but $(\omega_1^{-1} \circ \varphi') \circ (\pi_\lambda \times \pi_\mu) = (\pi_\lambda \otimes \pi_\mu) \circ \varphi$ and so ω_1 is bicontinuous. The bicontinuity of ω_2 follows even more easily.

Finally,

$$\operatorname{Hom}(B/I_\lambda \otimes_\pi B/I_\mu, \mathbf{C}) \underset{\text{(continuous)}}{\overset{\psi}{\hookrightarrow}} \operatorname{Hom}(B/I_\lambda \otimes_\pi B + B \otimes_\pi I_\mu, \mathbf{C})$$

and the latter is homeomorphic to $\operatorname{Hom}(B/I_\lambda + I_\mu, \mathbf{C}) = \operatorname{hull}(I_\lambda + I_\mu)$ [9]. ■

THEOREM 1. *If $\mathscr{B}$ is a fibre tensor product bundle, then* $\operatorname{Hom}(D, \mathbf{C})$ *may be identified with a kind of fibre bundle having base space* $\operatorname{Hom}(B, \mathbf{C})$, *fibre* $\operatorname{Hom}(A, \mathbf{C})$ *and group* $\operatorname{Auteo}(\operatorname{Hom}(A, \mathbf{C}))$.

PROOF. Since [8] $\operatorname{Hom}(B_\lambda \otimes_\pi A, \mathbf{C})$ may be identified with $\operatorname{Hom}(B_\lambda, \mathbf{C}) \times \operatorname{Hom}(A, \mathbf{C})$ and since $\operatorname{Hom}(B_\lambda, \mathbf{C}) = \operatorname{hull}(I_\lambda) = F_\lambda$ [9] there may be defined an equivalence relation $\sim_{\mathscr{B}}$ on the disjoint union $T = \dot{\bigcup}_\lambda F_\lambda \times \operatorname{Hom}(A, \mathbf{C})$ (* denoting dual):

$$(f_\lambda, a_\lambda) \sim_{\mathscr{B}} (f_\mu, a_\mu) \quad \text{iff} \quad \begin{cases} f_\lambda = f_\mu \in F_\lambda \cap F_\mu \\ \qquad \text{and} \\ a_\lambda = h_{\mu\lambda}^*(f_\mu)(a_\mu). \end{cases}$$

If $p_\lambda^*\colon F_\lambda \times \operatorname{Hom}(A, \mathbf{C}) \to \operatorname{Hom}(D, \mathbf{C})$ is the dual of $p_\lambda\colon D \to B_\lambda \otimes_\pi A$, there may be defined a map $\nu\colon T \to \operatorname{Hom}(D, \mathbf{C})$ so that $\nu|_{F_\lambda \times \operatorname{Hom}(A, \mathbf{C})} = p_\lambda^*$. It is directly verifiable that ν is continuous and compatible with $\sim_{\mathscr{B}}$, i.e., if $t_1, t_2 \in T$ and $t_1 \sim_{\mathscr{B}} t_2$, then $\nu(t_1) = \nu(t_2)$, whence ν defines a map

$$\vartheta\colon L \equiv T/\sim_{\mathscr{B}} \to \operatorname{Hom}(D, \mathbf{C}).$$

Thus, if π_λ is the composition

$$F_\lambda \times \operatorname{Hom}(A, \mathbf{C}) \overset{i_\lambda}{\hookrightarrow} T \overset{\pi}{\to} L \equiv T/\sim_{\mathscr{B}},$$

then $\vartheta \circ \pi_\lambda = p_\lambda^*$. The continuity of all maps considered with respect to the standard topologies is immediate. Furthermore, ϑ is injective as the following argument shows. [*Note*. Since $f_\lambda \in F_\lambda = \operatorname{hull}(I_\lambda)$, f_λ may be identified with an element, again denoted f_λ, of $\operatorname{Hom}(B_\lambda, \mathbf{C})$. Conversely, an element f_λ in $\operatorname{Hom}(B_\lambda, \mathbf{C})$ may be identified with an element of F_λ (in $\operatorname{Hom}(B, \mathbf{C})$).]

If $X = \pi_\lambda(f_\lambda, a_\lambda) \in L$, $Y = \pi_\mu(f'_\mu, a'_\mu) \in L$ and $\vartheta(X) = \vartheta(Y)$, then for $\tau = \{\tau_\lambda\}$ in D, $(f_\lambda \otimes a_\lambda)(\tau_\lambda) = (f'_\mu \otimes a'_\mu)(\tau_\mu)$. To show $X = Y$ it suffices to show $(f_\lambda, a_\lambda) \sim_{\mathscr{B}} (f'_\mu, a'_\mu)$. However, $f_\lambda = f'_\mu$ and their common value is in $\operatorname{hull}(I_\lambda) \cap \operatorname{hull}(I_\mu) = \operatorname{hull}(I_\lambda + I_\mu)$ iff for all x in $B/I_\lambda + I_\mu$, $\hat{x}(f_\lambda) = \hat{x}(f'_\mu)$. But if $g = (f_\lambda \otimes f'_\mu) \circ \omega_1 \circ \omega_2 \circ \omega_2 \circ \psi$, then Diagram 1 may be completed as follows:

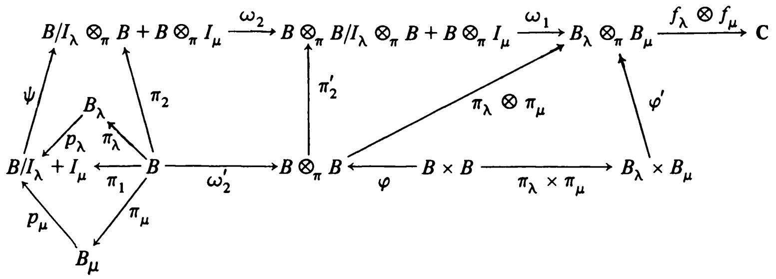

DIAGRAM 2

Thus, $f_\lambda = g \circ p_\lambda$ and $f'_\mu = g \circ p_\mu$ since if $z_\lambda \in B/I_\lambda$, then

$$\begin{aligned} f_\lambda(z_\lambda) &= (f_\lambda \otimes f'_\mu)(z_\lambda \otimes 1_{B_\mu}) = (f_\lambda \otimes f'_\mu)((\pi_\lambda \otimes \pi_\mu)(z \otimes 1_B) \quad \text{(for some } z \text{ in } B) \\ &= [(f_\lambda \otimes f'_\mu) \circ (\pi_\lambda \otimes \pi_\mu) \circ \omega'_2](z) = [(f_\lambda \otimes f'_\mu) \circ \omega_1 \circ \omega_2 \circ \pi_2](z) \\ &= (g \circ p_\lambda)(\pi_\lambda(z)) = (g \circ p_\lambda)(z_\lambda), \end{aligned}$$

i.e., $f_\lambda = g \circ p_\lambda$ and similarly $f'_\mu = g \circ p_\mu$. Since, for x in $B/I_\lambda + I_\mu$, $x = p_\lambda(x_\lambda) = p_\mu(x_\mu)$ it follows that $f_\lambda(x_\lambda) = f'_\mu(x_\mu)$, i.e., $\hat{x}(f_\lambda) = \hat{x}(f'_\mu)$, $f_\lambda = f'_\mu \equiv f \in \operatorname{hull}(I_\lambda) \cap \operatorname{hull}(I_\mu)$.

By definition $f \neq 0$. If $a_\mu \in A$, then $a_\lambda = h_{\lambda\mu}(f) a_\mu \in A$ and there is in $B/I_\lambda + I_\mu$ an x such that $f(x) = \hat{x}(f) \neq 0$. Thus, if $p_\lambda(x_\lambda) = p_\mu(x_\mu) = x$, then $x_\lambda/\hat{x}(f) \otimes a_\lambda \in B_\lambda \otimes A$, $x_\mu/\hat{x}(f) \otimes a_\mu \in B_\mu \otimes A$. Furthermore $(x_\lambda/\hat{x}(f))\hat{}\,(f) a_\lambda = a_\lambda$, $(\hat{x}_\mu(f)/\hat{x}(f)) h_{\lambda\mu}(f) a_\mu = (x_\mu/\hat{x}(f))\hat{}\,(f) h_{\lambda\mu}(f) a_\mu$ and so by direct calculation $a'_\mu = h^*_{\mu\lambda}(f) a_\lambda$. The definitions imply ϑ is also surjective and, since L is compact [**3**], ϑ is a homeomorphism. Furthermore, $\operatorname{Hom}(D, \mathbf{C})$ is a Hausdorff space [**8**]. Hence

$$L = \dot{\bigcup_\lambda} F_\lambda \times \operatorname{Hom}(A, \mathbf{C})/\sim_{\mathscr{B}} \underset{\text{homeo.}}{\cong} \operatorname{Hom}(D, \mathbf{C}).$$

Setting $\dot{\bigcup}_\lambda U_\lambda \times \operatorname{Hom}(A, \mathbf{C})/\sim$ to be the disjoint union (of the $U_\lambda \times \operatorname{Hom}(A, \mathbf{C})$) reduced by the equivalence relation $\sim$ derived from $\sim_{\mathscr{B}}$ ($\sim$ is the inverse image of

$\sim_{\mathscr{B}}$ under the canonical injection $\dot{\bigcup}_\lambda U_\lambda \times \mathrm{Hom}(A, \mathbf{C}) \hookrightarrow \dot{\bigcup}_\lambda F_\lambda \times \mathrm{Hom}(A, \mathbf{C}))$ yields the desired result. ∎

COROLLARY. *Let D' be the subalgebra consisting of all vectors $\tau = \{\tau_\lambda\}$ in S, $\tau_\lambda = \Sigma_i x_{\lambda i} \otimes a_{\lambda i}$ in $B_\lambda \otimes_\pi A$, and such that if $U_\lambda \cap U_\mu \neq \varnothing$ for some f in $U_\lambda \cap U_\mu$, $\hat{\tau}_\lambda(f) = \Sigma_i \hat{x}_{\lambda i}(f) a_{\lambda i} = \Sigma_j \hat{x}_{\mu j}(f) h_{\lambda\mu}(f) a_{\mu j}$. Then*

$$\mathrm{Hom}(D', \mathbf{C}) \underset{\text{homeo.}}{\cong} \dot{\bigcup_\lambda} U_\lambda \times \mathrm{Hom}(A, \mathbf{C})/\sim ,$$

$\sim$ *being the equivalence relation defined above.*

REMARK. The algebras D' and D differ only in the specifications $f \in U_\lambda \cap U_\mu$, and $f \in F_\lambda \cap F_\mu$, respectively.

PROOF. If i is the (continuous) injection $\dot{\bigcup}_\lambda U_\lambda \times \mathrm{Hom}(A, \mathbf{C})/\sim \hookrightarrow \dot{\bigcup}_\lambda F_\lambda \times \mathrm{Hom}(A, \mathbf{C})/\sim_{\mathscr{B}}$ and if $l = \vartheta \circ i(\vartheta\colon L \to \mathrm{Hom}(D, \mathbf{C})$ as defined earlier), then l maps $\dot{\bigcup}_\lambda U_\lambda \times \mathrm{Hom}(A, \mathbf{C})$ into $\mathrm{Hom}(D, \mathbf{C})$. Clearly there is an epimorphism $\xi\colon D \to D'$ and hence $\mathrm{Hom}(D', \mathbf{C})$ may be injected into $\mathrm{Hom}(D, \mathbf{C})$. It follows at once that $\mathrm{Image}(l) = \mathrm{Hom}(D', \mathbf{C})$, as required. ∎

3. Elaborations. It is of interest to explore the relation between the notions of a coordinate fibre tensor product bundle and a coordinate bundle in the usual sense. To this end let $\mathscr{E}$ be an arbitrary bundle with base space $\mathrm{Hom}(B, \mathbf{C})$, fibre $\mathrm{Hom}(A, \mathbf{C})$, and group $\mathscr{A}^*$ defined as in §2. Assume that $\mathscr{E}$ is specified by a coordinate bundle such that:

(i) *A is a locally convex algebra with identity and compact spectrum*; $\mathrm{Aut}(A)$ *is an equicontinuous subset of $\mathscr{L}_s(A)$.*

(ii) *B is a locally convex algebra with identity and for some finite open cover $\{V_i\}$ of* $\mathrm{Hom}(B, \mathbf{C})$ *each homeomorphism $\bar{\varphi}_{V_i}\colon V_i \times \mathrm{Hom}(A, \mathbf{C}) \to p^{-1}(V_i)$ has a unique extension to a fibre-preserving homeomorphism φ_{F_i} for some compact hull $F_i \supset \bar{V}_i$.*

(iii) *Each transition function $\bar{g}_{V_iV_j}\colon V_i \cap V_j \to \mathscr{A}^*$ has a unique continuous extension to a map $g_{F_iF_j}\colon F_i \cap F_j \to \mathscr{A}^*$ so that $\varphi_{F_i}(\beta, \alpha) = \varphi_{F_j}(\beta, g_{F_iF_j}(\beta)(\alpha))$.*

The situation above obtains, e.g., if A and B are the algebras of $\mathbf{C}$-valued $\mathscr{C}^\infty$-maps on compact differentiable manifolds **[9]**.

LEMMA 3. *Each coordinate fibre bundle $\mathscr{E}$ satisfying* (i), (ii) *and* (iii) *defines a coordinate fibre tensor product bundle.*

PROOF. Since $F_i = \mathrm{hull}(\mathrm{kernel}(F_i))$ **[9]**, if $I_i = \mathrm{kernel}(F_i)$ and $B_i = B/I_i$, then $\mathrm{Hom}(B_i, \mathbf{C})$ and F_i are homeomorphic. Since $\bar{g}_{V_iV_j}(f) \in \mathscr{A}^*$ it follows that there is a map $\bar{h}_{V_jV_i}\colon V_i \cap V_j \to \mathrm{Aut}(A)$ so that $\bar{g}_{V_iV_j}(f)$ is the dual of $\bar{h}_{V_jV_i}(f)$ for f in $V_i \cap V_j$. If $\mathrm{Aut}(A)$ is equipped with the compact open ("c") topology, then $\bar{h}_{V_iV_j}\colon V_i \cap V_j \to \mathrm{Aut}_c(A)$ is continuous as is $\bar{h}_{V_jV_i}\colon V_i \cap V_j \to \mathrm{Aut}_s(A)$. There are, corresponding to the maps $g_{F_iF_j}$, maps $h_{F_jF_i}$ and these serve as the transition maps that can be used to define the algebra S and the subalgebra D as in §1. (The ideals I_λ are now $\mathrm{kernel}(F_i)$.)

On the other hand, the maps $h_{F_jF_i}$ (and their restrictions $\bar{h}_{V_jV_i}$) can be used to define a "standard" locally convex algebra bundle $\tilde{\mathscr{E}}$ with base space $\mathrm{Hom}(B, \mathbf{C})$, fibre A, and group $\mathrm{Aut}(A)$. ∎

If $\Gamma(\tilde{\mathscr{E}})$ is the locally convex algebra of sections of $\tilde{\mathscr{E}}$, then there can be established an isomorphism

$$k\colon D \ni \{t_{V_i}\colon V_i \in \{V_i\}\} \mapsto \gamma(f) = \bar{\varphi}_{V_i}\left(f, \delta_{V_i f}(t_{V_i})\right) \in \Gamma(\mathscr{E}),$$

where $f \in V_i$ and $\delta_{V_i f}\colon B_i \otimes_\pi A \to A\colon \Sigma_m x_i^m \otimes a^m \mapsto \Sigma_m f(x_i^m) a^m$ is an "evaluation" map. The proof is outlined as follows:

(a) k is well defined and is independent of the choice of neighborhood containing f.

(b) k is 1-1 since $\bar{\varphi}_{V_i}$ is a homeomorphism and $\delta_{V_i f}$, as an evaluation map, is 1-1.

(c) k is surjective: If $\gamma \in \Gamma(\tilde{\mathscr{E}})$, $t_{V_i f}(\gamma(f)) \in A$ and thus $1_i \otimes t_{V_i f}(\gamma(f)) \in B_i \otimes A$ (1_i is the identity of $B_i = B/\text{kernel}(F_i)$). Hence the elements $\{1_i \otimes t_{V_i f}(\gamma(f))\} \in D$.

(d) k is bicontinuous if D is given the locally convex topology induced on it by $\Sigma_i B_i \otimes_\pi A$ and if $\Gamma(\tilde{\mathscr{E}})$ is given the topology described in [7]. Since [7] has not appeared at this writing, a brief description of the topology for $\Gamma(\tilde{\mathscr{E}})$ follows:

For V_i open in $\operatorname{Hom}(B, \mathbf{C})$ give the algebra $\Gamma(V_i, \tilde{\mathscr{E}})$ of sections over V_i seminorms

$$N_{K,p}(\gamma) = \sup_{x \in K} p\left(t_{V_i}(\gamma(x))\right),$$

K compact, $K \subset V_i$, p a seminorm from the set of seminorms that topologize A, t_{V_i} the usual isomorphism, $p^{-1}(x) = A$ [**5**].

If $i_{V_i}\colon \Gamma(\tilde{\mathscr{E}}) \to \Gamma(V_i, \mathscr{E})$ is the canonical continuous map ($\operatorname{Hom}(B, \mathbf{C}) = \bigcup_i V_i$), then with respect to the initial topology defined via $\{i_{V_i}\}$, $\Gamma(\tilde{\mathscr{E}})$ is a locally convex algebra of the type of A.

4. Equivalence classes. In analogy with the notion of equivalence classes of coordinate bundles [**10**] there can be defined an equivalence of coordinate fibre tensor product bundles. The corresponding equivalence classes define *fibre tensor product bundles*. The summarizing statement for these concepts and constructions is

THEOREM. *If A, B are locally convex algebras, each fibre bundle having base* $\operatorname{Hom}(B, \mathbf{C})$, *fibre* $\operatorname{Hom}(A, \mathbf{C})$ *and group $\mathscr{A}^*$ may be regarded as* $\operatorname{Hom}(D, \mathbf{C})$, *$D$ being a fibre tensor product bundle.* ■

REFERENCES

1. N. Bourbaki, *Theory of sets*, Addison-Wesley, Reading, Mass., 1968.

2. ______, *Topologie générale*, Chapitre X, Hermann, Paris, 1961.

3. J. Dugundji, *Topology*, Allyn & Bacon, Boston, Mass., 1966.

4. B. R. Gelbaum, *Tensor products over Banach algebras*, Trans. Amer. Math. Soc. **118** (1965), 131–149.

5. ______, *Banach algebra bundles*, Pacific J. Math. **28** (1969), 337–349.

6. A. Kryiazis, *On the spectra of topological* A*-tensor product* A*-algebras*, Yokohama Math. J. **31** (1983), 47–65.

7. ______, *Locally convex algebra bundles* (to appear).

8. A. Mallios, *On the spectrum of a topological tensor product of locally convex algebras*, Math. Ann. **154** (1964), 171–180.

9. ______, *General theory of topological algebras: selected topics* (to appear).

10. N. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, N. J., 1960.

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