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THE SOLUTION SETS OF EXTREMAL PROBLEMS IN H^1

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ABSTRACT. Let u be an essentially bounded function on the unit circle T. Let S_u denote the subset of the unit sphere of H^1 on which the functional $F \mapsto \int_0^{2\pi} \overline{u}(e^{it})F(e^{it})\,dt/2\pi$ attains its norm. A complete description of S_u is given in terms of an inner function b_0 and an outer function g_0 in H^2 for which g_0^2 is an exposed point in the unit ball of H^1. An explicit description is given for the kernel of an arbitrary Toeplitz operator on H^2. The exposed points in H^1 are characterized; an example is given of a strong outer function in H^1 which is not exposed.

1. Introduction. Let u be an essentially bounded function on the unit circle T, and let H^p denote the usual Hardy spaces on T for $p \geq 1$. If I_u is the functional on H^1 defined by

$$I_u F = \int_0^{2\pi} \overline{u}(e^{i\theta})F(e^{i\theta})\,d\theta/2\pi,$$

then an old problem (which will be solved in this article) is to parametrize the set $S_u = \{F \in S\colon I_u F = \|I_u\|\}$, where S denotes the unit sphere of H^1. This problem was solved in [**3**] by deLeeuw and Rudin for the special case that u is analytically continuable to $\{z\colon |z| > R\}$ for some $R < 1$. Nakazi [**6**] has also given a partial solution. If S_u is nonempty, then $S_u = S_{F/|F|}$ for some F in S. If $S_{F/|F|} = \{F\}$, F is said to be an exposed point of S (it is the unique point of contact that a certain hyperplane makes with the unit ball of H^1). A function G in H^1 will be called exposed if $S_{G/|G|}$ is a singleton set. It has been conjectured that a function F is exposed if and only if it is a strong outer function, i.e., if and only if F cannot be factored in the form $F(z) = (z-a)^2G(z)$, where G is in H^1 and a belongs to T (see [**3** or **5**]). An example will be furnished in §6 to show that a strong outer function need not be exposed. A characterization of the exposed points in H^1 will also be given, though the problem of finding an "effective" characterization of the exposed points still remains open.

§2 contains the central results of this paper: a concrete analysis of the spaces $M_k = z^k H^2 \cap (h/\overline{h})\overline{H}^2$ is given, where h denotes an outer function in H^2 and the bar denotes complex conjugation. These spaces arise naturally in the study of stationary stochastic processes (see [**2**]). Most of the other results in this paper follow from Theorem 3.

In §3 it is shown that the kernel of an arbitrary Toeplitz operator can be expressed as the L^2 closure of $L^2 \cap g(H^2 \ominus bH^2)$, where g^2 is exposed in H^1 and b is an inner function.

§4 translates the preceding result into the language of Hankel operators.

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§5 contains a parametrization for S_u and also sheds a little light on the structure of such sets.

§6 contains some remarks about exposed points of H^1. In particular, it is shown that strong outer functions need not be exposed, and a characterization of exposed points is given.

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2. The spaces M_k. Let h be an outer function in H^2. Define M_k as before and let M_k' denote $\bar{z}^k M_k$. It was shown in [**2**] that $\dim(M_k/M_{k+1}) \leq 1$ and $M_k = M_{k+1}$ if and only if $M_k = \{0\}$. The following theorem was also proven (using a slightly different definition of M_k).

THEOREM 1. *Let h be outer in H^2. Then the following are equivalent:*

(2.1) $M_k \neq \{0\} = M_{k+1}$.

(2.2) $|h|^2 = |P|^2|g|^2$, *where g^2 is exposed in H^1 and P is a polynomial of degree k with all of its roots on T.*

(2.3) $h/\bar{h} = z^k g/\bar{g}$, *where g^2 is exposed in H^1.*

(2.4) $h/\bar{h} = bg/\bar{g}$, *where g^2 is exposed in H^1 and b is a Blaschke product of order k.*

It was also noted in [**2**] that h^2 fails to be exposed in H^1 if and only if $h/\bar{h} = zbg/\bar{g}$ for some inner function b and g outer in H^2.

Now, for each k such that $M_k \neq \{0\}$, choose r_k in the unit sphere of M_k to maximize the functional $r \mapsto \operatorname{Re}\langle r, z^k\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $L^2(T, d\theta/2\pi)$. Thus, r_k is a real scalar multiple of the projection of z^k on M_k. A moment's thought reveals that $r_k = z^k g_k$, where g_k is an outer function in M_k'. Note that $z^k g_k = (h/\bar{h})\bar{b}_k\bar{g}_k$, where b_k is some inner function. We then have the following result.

THEOREM 2. *If $M_k \neq \{0\}$, each of the following is true:*

(2.5) $z^k g_k \in M_k \ominus M_{k+1}$.

(2.6) $b_k g_k \in M_k' \ominus M_{k+1}'$.

(2.7) g_k^2 *is exposed in* H^1.

PROOF. First, note that if $z^{k+1}f \in M_{k+1}$ and P_{M_k} denotes orthogonal projection on M_k, then

$$\langle z^k g_k, z^{k+1} f\rangle = g_k(0)^{-1}\langle P_{M_k} z^k, z^{k+1} f\rangle = g_k(0)^{-1}\langle z^k, z^{k+1} f\rangle = 0.$$

Thus, (2.5) holds. Item (2.6) follows from (2.5) by noting that $b_k g_k = (h/\bar{h})\bar{z}^k\bar{g}_k$.

Now assume (2.7) fails. Then we may write $g_k/\bar{g}_k = zbf/\bar{f}$, where b is inner and f is outer in H^2. Replacing f by $(1+b)f$, we may assume, without loss of generality, that $g_k/\bar{g}_k = zf/\bar{f}$, where f is outer. Thus $z^{k+1}f \in M_{k+1}$. Now $\langle g_k, f\rangle = \langle zf, g_k\rangle$ and both quantities equal zero by (2.5). Set

$$s_t = z^k g_k + t z^k \overline{f(0)} f,$$

where t is a real parameter. Then $s_t \in M_k$ and

$$\frac{\langle s_t, z^k\rangle}{\|s_t\|_2} = \frac{\langle r_k, z^k\rangle + t|f(0)|^2}{[1 + t^2|f(0)|^2\|f\|_2^2]^{1/2}}. \tag{2.8}$$

For small positive t the extremal property of r_k is contradicted by (2.8), so we must have g_k^2 exposed in H^1, and Theorem 2 is proved.

It turns out that b_k and g_k can be defined recursively from b_0 and g_0. Let $a_k = b_k(0)$.

THEOREM 3. *For each k such that $M_{k+1} \neq \{0\}$, there exists a $\lambda_{k+1} \in C$ such that*

(1) $g_{k+1} = \lambda_{k+1} g_k \cdot (1 - \bar{a}_k b_k)$,
(2) $b_{k+1} = (b_k - a_k)/z(1 - \bar{a}_k b_k)$.

PROOF. Write $\psi_{k+1} = (b_k - a_k)/z(1 - \bar{a}_k b_k)$, and $f_{k+1} = g_k \cdot (1 - \bar{a}_k b_k)$. Then ψ_{k+1} is an inner function and f_{k+1} is outer. Using the relation $h/\bar{h} = z^k b_k g_k / \bar{g}_k$, we have

$$\begin{aligned} z^{k+1}\psi_{k+1} f_{k+1}/\bar{f}_{k+1} &= z^k \frac{b_k - a_k}{1 - \bar{a}_k b_k} \cdot \frac{g_k}{\bar{g}_k} \cdot \frac{1 - \bar{a}_k b_k}{1 - a_k \bar{b}_k} \cdot \frac{b_k}{b_k} \\ &= z^k b_k g_k / \bar{g}_k = h/\bar{h}. \end{aligned}$$

Hence, $z^{k+1} f_{k+1} \in M_{k+1}$. Now, let $\phi = h/\bar{h}$. Then for $n > k+1$,

$$\begin{aligned} \langle z^{k+1} f_{k+1}, z^n g_n \rangle &= \langle z^{k+1} g_k, z^n g_n \rangle - \bar{a}_k \langle z^{k+1} b_k g_k, z^n g_n \rangle \\ &= \langle z^k g_k, z^{n-1} g_n \rangle - \bar{a}_k \langle z^{k+1} b_k g_k, z^n g_n \rangle \\ &= 0 - \bar{a}_k \langle z\phi \bar{g}_k, \bar{b}_n \bar{g}_n \phi \rangle = -\bar{a}_k \langle z^{k+1} b_n g_n, z^k g_k \rangle. \end{aligned}$$

Now, $z^{k+1} b_n g_n \in M_{k+1}$, so this last inner product vanishes. By Theorem 2 we then have $z^{k+1} f_{k+1} = \lambda_{k+1} z^{k+1} g_k$ for some $\lambda_{k+1} \in C$. Now, it is easily checked that $b_{k+1} = \psi_{k+1}$.

COROLLARY 4. *$L^2 \cap g_0(H^2 \ominus z b_0 H^2)$ is a dense subset of M_0.*

PROOF. Fix h, b_k, g_k and a_k as above. Let $h_0 = 1 + b_0$, $m_k = z^k H^2 \cap (h_0/\bar{h}_0)\overline{H}^2$, and $m_k' = \bar{z}^k m_k$. Carrying out the program of Theorem 2, let $\{z^k G_k\}$ be the o.n. basis for m_0 with $z^k G_k \in m_k \ominus m_{k+1}$. Then $G_0 = 1$ and $b_0 = h_0/\bar{h}_0 = b_0 G_0/\overline{G}_0$, and for $k \geq 1$, $h_0/\bar{h}_0 = z^k B_k G_k/\overline{G}_k$, where B_k is inner. The proof of Proposition 4 shows that $B_k = b_k$ for each k. Thus, $\beta = \{1, z(1 - \bar{a}_0 b_0), z^2(1 - \bar{a}_1 b_1), \ldots\}$ forms an orthogonal basis for $m_0 = H^2 \cap b_0 \overline{H}^2 = H \ominus z b_0 H^2$. But $g_0 \beta = \{g_0, z g_0(1 - \bar{a}_0 b_0), \ldots\}$, which is a spanning set for M_0. Also, if an L^2 function g belongs to $g_0(H^2 \cap b_0 \overline{H}^2)$, it clearly belongs to M_0. Thus M_0 is the L^2 closure of $L^2 \cap g_0(H^2 \ominus z b_0 H^2)$.

REMARK. It is easily seen that M_0 is finite dimensional if and only if b_0 is a finite Blaschke product, in which case the dimension of M_0 equals one plus the order of b_0. The construction of the g_k provides a simple way to obtain an orthogonal basis for $H^2 \ominus b H^2$ in the case where b is an inner function which is not a Blaschke product.

3. The kernels of Toeplitz operators. Let f be an essentially bounded function on T and let T_f denote the Toeplitz operator on H^2 defined by $T_f g = P(fg)$, where P denotes orthogonal projection from L^2 onto H^2. If $f = \bar{b}$, where b is an inner function, then $\operatorname{Ker} T_f$, the kernel of T_f, is just $H^2 \ominus b H^2$ and projection onto this space is easily carried out. It turns out that, in general, $\operatorname{Ker} T_f$ is a weighted version of the above. This was shown independently by Nakazi [**6**] for the case that $\operatorname{Ker} T_f$ is finite dimensional.

LEMMA 5. *Let $f \in L^\infty$ and suppose $\operatorname{Ker}(T_f) \neq \{0\}$. Then there exists an outer function h in H^2 such that $\operatorname{Ker}(T_f) = \operatorname{Ker}(T_{\overline{h}/h})$.*

PROOF. Let g be a nontrivial function in $\operatorname{Ker}(T_f)$. Then there exists a $k \in H^2$ such that $fg = \overline{zk}$ (z denotes the identity function of T). Thus, $|f| = |k/g|$, so $\log|f|$ is integrable, hence we may write $f = uF$, where F is outer in H^∞ and $|u| = 1$ a.e. on T. Now, if $g \in H^2$, write $g = BG$, where G is outer and B is inner. Then $T_f g = 0$ iff $uFBG = \overline{zB_1 F\, G}$ for some inner function B_1. Thus, $T_f g = 0$ if and only if $u(F/\overline{F})g$ is in $(H^2)^\perp$, i.e., if and only if $T_{uF/\overline{F}}g = 0$. Note also that

$$uF/\overline{F} = \overline{z}\overline{B}\,\overline{B}_1\overline{G}/G = \overline{(1 + zBB_1)G}/(1 + zBB_1)G.$$

Now, let $h = (1 + zBB_1)G$. The first factor takes its values in the right half-plane and is bounded, hence it is outer in H^∞. Thus, h is seen to be outer in H^2 with

$$\operatorname{Ker} T_f = \operatorname{Ker} T_{uF/\overline{F}} = \operatorname{Ker}(T_{\overline{h}/h}).$$

This proves the lemma.

THEOREM 6. *Suppose f is not identically zero and $\operatorname{Ker} T_f$ is nontrivial. Then there is an outer function h, an inner function b_1, and an outer function g_1 whose square is exposed in H^1 such that $L^2 \cap g_1(H^2 \ominus zb_1H^2)$ is dense in $\operatorname{Ker} T_f$. In fact, an orthogonal basis for $\operatorname{Ker} T_f$ is given by*

$$\{g_1, z(1 - \overline{a}_1 b_1)g_1, z^2(1 - \overline{a}_1 b_1)(1 - \overline{a}_2 b_2)g_1, \ldots\},$$

where a_k, b_k, and g_k are related to h as in §2.

PROOF. From the previous lemma we may assume that $\operatorname{Ker} T_f = \operatorname{Ker} T_{\overline{h}/h}$, where h is outer in H^2. It is easily checked that this last kernel is $H^2 \cap \overline{z}(h/\overline{h})\overline{H}^2 = M_1'$. This is spanned by the set $\{g_1, zg_2, z^2g_3, \ldots\}$, which, by Theorem 3, spans the same space as does

$$g_1\{1, z(1 - \overline{a}_1 b_1), z^2(1 - \overline{a}_1 b_1)(1 - \overline{a}_2 b_2), \ldots\}.$$

This is contained in $L^2 \cap g_1(H^2 \ominus zb_1H^2)$ which, in turn, is a subset of M_1'.

4. Hankel operators. For an essentially bounded function f on T, let H_f denote the Hankel operator from H^2 into $(H^2)^\perp$ defined by $H_f g = (I - P)(fg)$, where I is the identity operator on L^2. Then by a theorem of Nehari,

$$\|H_f\| = \inf\{\|f - g\|_\infty : g \in H^\infty\}$$

(see [**7**]). Using a normal families argument, we may assume, without loss of generality, that $\|H_f\| = \|f\|_\infty$. Let $N = \{g \in H^2\colon \|H_f g\|_2 = \|H_f\| \cdot \|g\|_2\}$. If N contains a nontrivial function g, then we have

$$\|f\|_\infty\|g\|_2 = \|H_f g\|_2 \leq \|fg\|_2 \leq \|f\|_\infty\|g\|_2,$$

so $|f| = \|f\|_\infty$ a.e. on T. To describe N in the case that $N \neq \{0\}$, we then assume, without loss of generality, that $|f| = 1$ a.e. on T. Then

$$N = \operatorname{Ker}(I - H_f^* H_f)^{1/2} = \operatorname{Ker}(T_f^* T_f)^{1/2} = \operatorname{Ker}(T_f^* T_f) = \operatorname{Ker} T_f.$$

By Lemma 5 we may write $f = h/\overline{h}$ for some outer function h in H^2, so N is of the form in Theorem 6.

5. A parametrization of S_u. Let S_u be defined as in §1. If S_u is nonempty, then $S_u = S_{F/|F|}$ for some function F in H^1. As before, we may assume, without loss of generality, that F is outer. Let $h = F^{1/2}$. Now form the spaces M_k as in §2 along with b_k and g_k.

THEOREM 7. *Let S_u be nonempty. If h, b_k, and g_k are as above, then every G in S_u is of the form $G = (h/\overline{h})|g|^2$, where $\|g\|_2 = 1$ and g is in the L^2 closure of $L^2 \cap g_0(H^2 \ominus zb_0H^2)$.*

PROOF. We have $F/|F| = h/\overline{h} = b_0g_0/\overline{g}_0$ in the notation of §2. If $G \in S_{F/|F|}$, then write $G = Bg^2$, where g is outer in H^2 and B is inner. then $G/|G| = Bg/\overline{g} = h/\overline{h}$, so Bg is seen to be in $H^2 \cap (h/\overline{h})\overline{H}^2 = M_0$. Thus,

$$G = (h/\overline{h})|G| = (h/\overline{h})|Bg|^2.$$

Conversely, if $Bg \in M_0$, where B is inner and g is outer with unit norm in H^2, then $Bg = (h/\overline{h})\overline{B_1g}$, where B_1 is inner, so $(h/\overline{h})|Bg|^2 = BB_1g^2$ and

$$\operatorname{Arg}(BB_1g^2) = \operatorname{Arg}(h/\overline{h}) = \operatorname{Arg}(F/|F|),$$

so $(h/\overline{h})|Bg|^2$ belongs to $S_{F/|F|}$. The theorem now follows from Corollary 4.

It was noted in [**2**] that if $S_{F/|F|}$ contains a strong outer function and is not a singleton, then it contains functions with arbitrarily many zeroes in the unit disk D. This is a direct consequence of Theorem 1. It was also conjectured that, in this case, $S_{F/|F|}$ must contain a function whose inner part is not a finite Blaschke product. Corollary 4 provides an affirmative answer to this conjecture. To see this, note that if there is no bound for the number of zeroes for functions in $S_{F/|F|}$, then the related space M_0 must be infinite dimensional. Hence b_0 is not a finite Blaschke product, yet $b_0g_0^2$ is in $S_{F/|F|}$.

6. Exposed points in H^1. It was noted by deLeeuw and Rudin in [**3**] that every exposed point F in H^1 is a strong outer function. It was conjectured by Nakazi [**6**] that the converse also holds. Theorem 1 provides some evidence of this. Other sufficient conditions for F to be exposed are discussed in [**2**]. An easy way to construct an outer function in H^1 which is not exposed is to take $F = (1+B)^2$, where B is an inner function. F is outer since $1+B$ takes its values in the right half-plane, and $F/|F| = B$, so F is not exposed. Consider a Blaschke product B with zero sequence $\{w_n\}$. It was shown by Ahern and Clark in [**1**] that $1+B = (z-a)g$ for some g in H^2 and $a \in T$ only if

$$\sum_{n=1}^{\infty} \frac{1-|w_n|^2}{|1-\bar{a}w_n|^2} < \infty. \tag{6.1}$$

However, there is an example (cited in [**1**]) due to Frostman [**4**] of a Blaschke product for which the sum in (6.1) diverges for every $a \in T$. It then follows that for this choice of B, $(1+B)^2$ is a strong outer function which is not exposed. Hence the conjecture is false.

It would be of interest to have a usable characterization of the exposed points of H^1. The next theorem gives a characterization, though its usability is questionable.

THEOREM 8. *Let h be an outer function in the unit sphere of H^2. Then h^2 is not exposed if and only if there exists a positive constant $C < 1$ such that*

$$|h(0)| \leq C\|T_{h/\bar{h}}(1 + zf)\|_2 \tag{6.2}$$

for every $f \in H^2$.

PROOF. Using the notation of §2, note that $h \in M_0'$ and h^2 is exposed if and only if $M_1 = M_1' = \{0\}$. Now, $M_1' = \operatorname{Ker} T_{\bar{h}/h} = (\operatorname{Range} T_{h/\bar{h}})^{\perp}$. (See [**7**] for a discussion of Toeplitz operators.) Thus, if $M_1' = \{0\}$, $h \in (M_1')^{\perp}$, so h is in the closure of the range of $T_{h/\bar{h}}$. In this case,

$$\begin{aligned} 1 = \|h\|_2 &= \sup\{|\langle h, T_{h/\bar{h}}g\rangle|/\|T_{h/\bar{h}}g\|_2\} \\ &= \sup\{|\langle h, (h/\bar{h})g\rangle|/\|T_{h/\bar{h}}g\|_2\} = \sup\{|h(0)| \cdot |g(0)|/\|T_{h/\bar{h}}g\|_2\}, \end{aligned}$$

where the supremum is taken over all $g \in H^2$ such that $T_{h/\bar{h}}g \neq 0$. Note that

$$\langle h, T_{h/\bar{h}}g\rangle = \overline{h(0)} \cdot \overline{g(0)},$$

so if $g(0) \neq 0$, then $T_{h/\bar{h}}g \neq 0$. Hence, (6.2) can hold only if $M_1' \neq \{0\}$.

Suppose, conversely, that (6.2) fails to hold. Then h is orthogonal to M_1' so, by Theorem 2, h is a scalar multiple of b_0g_0. But since h is outer, we must have that b_0 is a unimodular constant, and hence, by the same theorem, h^2 must be an exposed point of H^1. This establishes the theorem.

Finally, it seems that the key to characterizing the exposed points of H^1 lies in understanding the behavior of bounded outer functions in the left invariant subspaces $H^2 \ominus bH^2$ for arbitrary inner functions b. For, suppose that F is outer in S and is not exposed. Take any other outer function G in $S_{F/|F|}$. We then have $(F+G)/2 = BK \in S_{F/|F|}$, where B is a nonconstant inner function and K is outer (this is because $(F+G)/2$ is not an extreme point of S: see [**3**]). Note that $|K| \geq |G|/2$ and $|K| \geq |F|/2$ a.e. on T since F and G have the same argument a.e. Thus, $F = 2K(B - G/2K)$. Now $\operatorname{Arg}(G/2K) = \operatorname{Arg}(B)$ a.e., and $|G/2K| < 1$ a.e., and $B - G/2K$ is outer since F is. Note further that $|B - G/2K| = 1 - |G/2K|$. Thus we may write $F = 2Kg^2$, where g is outer in $H^2 \ominus zBH^2$ and $0 < |g| < 1$ a.e. on T. Of course, when B turns out to be a finite Blaschke product, g^2 has double zeroes on T by Theorem 1.

ADDED IN PROOF. The author can now show that $M_0 = g_0(H^2 \ominus zb_0H^2)$.

REFERENCES

1. P. R. Ahern and D. N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. **92** (1970), 332–342.
2. P. B. Bloomfield, N. P. Jewell and E. Hayashi, *Characterizations of completely nondeterministic stochastic processes*, Pacific J. Math. **107** (1983), 307–317.
3. K. deLeeuw and W. Rudin, *Extreme points and extreme problems in H^1*, Pacific J. Math. **8** (1958), 467–485.
4. O. Frostman, *Sur les produits de Blaschke*, Kungl. Fysiografiska Säliskakets i Lund Förhandlingar **12** (1942), 169–182.

5. T. Nakazi, *Exposed points in H^1*, preprint.
6. ——, *The kernels of Toeplitz operators*, preprint.
7. D. E. Sarason, *Function theory on the unit circle*, Virginia Polytechnic Inst. and State Univ., Blacksburg, Va., 1979.

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