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A NOTE ON A WEIGHTED SOBOLEV INEQUALITY

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ABSTRACT. We give a simple proof of a weighted imebedding theorem whose proof was originally given in [**3**].

The purpose of this note is to provide a simplified proof of a weighted imbedding theorem previously proved by Fabes, Kenig and Serapioni [**3**]. They proved the following inequality (see Theorem (1.2) in [**3**]),

$$(1)\quad \left(\frac{1}{w(B_R)}\int_{B_R}|u(x)|^{kp}w(x)\,dx\right)^{1/kp}\le cR\left(\frac{1}{w(B_R)}\int_{B_R}|\nabla u|^p w(x)\,dx\right)^{1/p}.$$

Here $w\in A_p$ $(1<p<+\infty)$, $1\le k\le n/(n-1)+\delta$ and u is any function in $C_0^\infty(B_R)$ (henceforth $w(B_R)=\int_{B_R}w\,dx$ and n is the dimension).

Our short proof, inspired by an idea in [**4**], is an easy consequence of the boundedness of the Hardy-Littlewood maximal operator Mf between weighted L^p spaces iff the weight is A_p (see [**5**, **2**] and, for an elementary proof, the recent paper [**1**]). In the following assume the reader is familiar with the relevant definitions and notations as given in [**3**].

Set $If(x)=\int_{R^n}|f(y)|\,|x-y|^{1-n}\,dy$. It is well known that (1) is an immediate consequence of the following inequality:

$$(2)\quad \left(\frac{1}{w(B_R)}\int_{B_R}[If(x)]^{kp}w(x)\,dx\right)^{1/kp}\le cR\left(\frac{1}{w(B_R)}\int_{B_R}|f(x)|^p w(x)\,dx\right)^{1/p}.$$

Here $w\in A_p$ $(1<p<+\infty)$, $1\le k\le n/(n-1)+\delta$, $f\in L^p(B_R,w)$, and c,δ are positive constants independent on f and R.

To prove (2) set, for any $\varepsilon>0$,

$$I^{(\varepsilon)}f(x)=\int_{|x-y|\le\varepsilon}|f(y)|\,|x-y|^{1-n}\,dy.$$

It is easy to see that $I^{(\varepsilon)}f(x)\le c\varepsilon Mf(x)$. Further,

$$\begin{aligned}&If(x)-I^{(\varepsilon)}f(x)\equiv I_{(\varepsilon)}f(x)\\ (3)\qquad &\le\|f\|_{L^p(B_R,w)}\left(\int_{\{|x-y|>\varepsilon\}\cap B_R}|x-y|^{(1-n)p'}w^{-1/(p-1)}\,dy\right)^{1/p'},\\ &\left(\frac{1}{p}+\frac{1}{p'}=1\right).\end{aligned}$$

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Because $w \in A_p$, a q can be chosen such that $w \in A_q$, $1 < q < p$, $n - p/q > 0$ (see [**2**]). Hence,

$$I_{(\varepsilon)} f(x) \leq c\|f\|_{L^p(B_R,w)} \left(\int_{B_R} [w(y)]^{-1/(q-1)}\, dy\right)^{(q-1)/p} \varepsilon^{1-nq/p}. \tag{4}$$

(3) and (4) imply

$$If(x) \leq c\varepsilon Mf(x) + c\|f\|_{L^p(B_R,w)} \left(\int_{B_R} [w(y)]^{-1/(q-1)}\, dy\right)^{(q-1)/p} \varepsilon^{1-nq/p}. \tag{5}$$

We minimize with respect to ε the right side of (5) to get

$$If(x) \leq c[Mf(x)]^{1-p/nq} \|f\|^{p/nq}_{L^p(B_R,w)} \left(\int_{B_R} [w(y)]^{-1/(q-1)}\, dy\right)^{(q-1)/nq},$$

and, using the boundedness of Mf in the $L^p(B_R, w)$ norm,

$$\|If(x)\|_{L^{pk}(B_R,w)} \leq c\|f\|_{L^p(B_R,w)} \left(\int_{B_R} [w(y)]^{-1/(q-1)}\, dy\right)^{(q-1)/nq},$$

where $k = nq/(nq-p)$. To complete the proof we divide by $[w(B_R)]^{(nq-p)/npq}$ and use the A_q condition.

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