# POWERS OF TRANSITIVE BASES OF MEASURE AND CATEGORY 

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#### Abstract

We prove that on the real line the minimal cardinality of a base of measure zero sets equals the minimal cardinality of their transitive base. Next we show that it is relatively consistent that the minimal cardinality of a base of meager sets is greater than the minimal cardinality of their transitive base. We also prove that it is relatively consistent that the transitive additivity of measure zero sets is greater than the ordinary additivity and that the same is true about meager sets.


0. Introduction. Let $\mathbf{P}$ be any group and $\mathscr{I}$ any ideal of subsets of $\mathbf{P}$. We call a family $\mathscr{A} \subseteq \mathscr{I}$ a base of $\mathscr{I}$ if for each $I \in \mathscr{I}$ there is an $A \in \mathscr{A}$ such that $I \subseteq A$. The minimal cardinality of a base of $\mathscr{I}$ is denoted by $\Delta(\mathscr{I})$. A family $\mathscr{A} \subseteq \mathscr{I}$ is called a transitive base of $\mathscr{I}$ if for each $I \in \mathscr{I}$ there exist $A \in \mathscr{A}$ and $p \in \mathbf{P}$ such that $p \cdot I \subseteq A$. The minimal cardinality of a transitive base of $\mathscr{I}$ is denoted by $\Delta_{t}(\mathscr{I})$. The additivity of $\mathscr{I}$, denoted by $\operatorname{add}(\mathscr{I})$, is the minimal cardinality of a family $\mathscr{A} \subseteq \mathscr{I}$ such that $\cup \mathscr{A} \notin \mathscr{I}$. The transitive additivity, denoted by $\operatorname{add}_{t}(\mathscr{I})$, is the minimal cardinality of a set $X \subseteq \mathbf{P}$ such that $X \cdot I \notin \mathscr{I}$ for some $I \in \mathscr{I}$.

Our set theory is ZFC. We use the standard set theoretical notation. In particular, $\omega$ is the set of all natural numbers, $\mathscr{P}(\omega)$ is the set of all subsets of $\omega$ and ${ }^{\omega} \omega$ is the set of all functions from $\omega$ to $\omega$. An ordinal is the set of proceeding ordinals, e.g. if $n$ is a natural number, then $n=\{0,1, \ldots, n-1\}$. If $X$ is any set and $\kappa$ any cardinal, then let $|X|$ denote the cardinality of $X,[X]^{\kappa}$ the set of all subsets of $X$ of cardinality $\kappa$ and $[X]^{\leqslant \kappa}$ the set of all subsets of $X$ of cardinality not greater than $\kappa$. Let $\mathbf{R}$ denote the real line with its usual topology and algebraic structure. Let $\mathbf{Q}$ stand for rationals and $\mathbf{Z}$ for integers. We fix some enumeration $\left\{q_{n}: n \in \omega\right\}$ of $\mathbf{Q}$. Let $\mu$ denote the Lebesque measure on $\mathbf{R}$. If $p, r \in \mathbf{R}$ and $p \leqslant r$, then let $[p ; r$ ] denote the closed interval and ( $p ; r$ ) the open interval with endpoints $p, r$. Let $\exists n^{\infty}$ mean "there is infinitely many natural $n$ " and $\forall n^{\infty}$ mean "for all but finitely many natural $n$." For $f, g \in{ }^{\omega} \omega$ we write $f<g$ if for any $n<\omega$, we have $f(n)<g(n)$ and we write $f<g$ if $\forall n^{\infty} f(n)<g(n)$. A subset $D \subseteq{ }^{\omega} \omega$ is a dominating family if for any function $f \in{ }^{\omega} \omega$ there is $d \in D$ with $f<d$. The minimal cardinality of a dominating family is denoted by $\Delta$. A subset $A \subseteq{ }^{\omega} \omega$ is unbounded if there is no $f \in{ }^{\omega} \omega$ such that for any $g \in A$ we have $g \prec f$. The minimal cardinality of an unbounded family is denoted by $\lambda$.

[^0]In this paper we deal with the ideal $\mathscr{L}$ of null sets (measure zero sets) of $\mathbf{R}$ and the ideal $\mathscr{K}$ of meager sets (first category sets) of $\mathbf{R}$. We show that $\Delta_{t}(\mathscr{L})=\Delta(\mathscr{L})$ and $\Delta_{t}(\mathscr{K})=\Delta$. Since models of set theory are known where $\Delta(\mathscr{K})>\Delta$, this result establishes a rather surprising difference between $\mathscr{K}$ and $\mathscr{L}$. We also construct a model of set theory in which $\operatorname{add}(\mathscr{L})<\operatorname{add}_{t}(\mathscr{L})$ and $\operatorname{add}(\mathscr{K})<\operatorname{add}_{t}(\mathscr{K})$. In our model we also have $\operatorname{add}(\mathscr{L})<\operatorname{add}(\mathscr{C})$, where $\mathscr{C}$ is the ideal of strong measure zero subsets of $\mathbf{R}$ (i.e. $A \in \mathscr{C}$ iff for each sequence $\left\langle\varepsilon_{i}\right\rangle_{i<\omega}$ of positive reals there exists a sequence $\left\langle I_{i}\right\rangle_{i<\omega}$ of intervals such that for any $i<\omega$, length $\left(I_{i}\right)<\varepsilon_{i}$ and $\left.A \subseteq \cup_{I} I_{i}\right)$. Finally we show that the equality $\Delta_{t}(\mathscr{K})=\Delta$ can be extended to any Polish group and the equality $\Delta_{t}(\mathscr{L})=\Delta(\mathscr{L})$ to any abelian locally compact Polish group.

We use the following technique due to T. Bartoszyński [B]. Suppose we are given a function $f \in{ }^{\omega}(\omega \backslash 1)$. Let $\Pi_{n} f(n)$ denote the infinite cartesian product of sets $f(0), f(1), \ldots$ and $\Omega$ be the mapping from $\Pi_{n} f(n)$ onto the unit interval $[0 ; 1]$ defined for each $g \in \Pi_{n} f(n)$ by $\Omega(g)=\sum_{n} g(n)(f(0) \cdot \cdots \cdot f(n))^{-1}$. For each $a \in$ $\Pi_{n} \mathscr{P}(f(n))$ let $\bar{a}=\left\{g \in \Pi_{n} f(n): \exists n^{\infty} g(n) \in a(n)\right\}$. It is easy to see that $\Omega^{\prime \prime} \bar{a} \in$ $\mathscr{L}$ iff $\sum_{n}|a(n)| f(n)^{-1}<\infty$. Let $\mathscr{A}_{f}=\left\{a \in \Pi_{n} \mathscr{P}(f(n)): \sum_{n}|a(n)| f(n)^{-1}<\infty\right\}$. Note that if $g \in \Pi_{n} f(n)$, then also $g \in \Pi_{n} \mathscr{P}(f(n))$.

The following lemma is motivated by $[B]$.
Lemma 0.1. Let $A \subseteq[0,1]$ be a null set. Then there is an $a \in \mathscr{A}_{f}$ such that for any $b \in \mathscr{A}_{f}$ if $\Omega^{\prime \prime} \bar{b} \subseteq A$, then $\forall n^{\infty} b(n) \subseteq a(n)$.

Proof. There exists a perfect tree $T \subseteq \bigcup_{m} \Pi_{n<m} f(n)$ such that for the set of its branches [ $T$ ] we have: $\Omega^{\prime \prime}[T]$ is not null and is disjoint with $A$. For each $\tau \in T$ let $T_{\tau}=\{\sigma \in T: \sigma \subseteq \tau \vee \sigma \subseteq \tau\}$. We can assume that for each $\tau \in T$ the set $\Omega^{\prime \prime}\left[T_{\tau}\right]$ is not null, otherwise we can replace $T$ by the tree $T \backslash\left\{\tau: \Omega^{\prime \prime}\left[T_{\tau}\right] \in \mathscr{L}\right\}$. For any $\tau \in T$ and $n<\omega$ we set $T_{\tau}(n)=\left\{t(n): t \in\left[T_{\tau}\right]\right\}$ and $a_{\tau}(n)=f(n) \backslash T_{\tau}(n)$. Then for each $b \in \mathscr{A}_{f}$ such that $\Omega^{\prime \prime} \bar{b} \subseteq A$, there exists $\tau \in T$ such that $\forall n^{\infty} b(n) \subseteq a_{\tau}(n)$. To see this let $b$ be given and suppose not. Then for any $\tau \in T$ we have $\exists n^{\infty} b(n) \cap T_{\tau}(n)$ $\neq \varnothing$. We define a sequence $\left\langle\tau_{m}\right\rangle_{m<\omega}$ : let $\tau_{0}=\varnothing$; suppose that $\tau_{m}$ is defined, that there is an $n>$ length $\left(\tau_{m}\right)$ such that $b(n) \cap T_{\tau_{m}}(n) \neq \varnothing$, and let $\tau_{m+1}$ be such that $\tau_{m+1} \supseteq \tau_{m}$ and $\tau_{m+1}(n) \in b(n) \cap T_{\tau_{m}}(n)$. Let $t=\bigcup_{m} \tau_{m}$. Then $\tau \in[T]$ and $\exists n^{\infty}$ $t(n) \in b(n)$. So $\Omega(t) \in \Omega^{\prime \prime} \bar{b} \subseteq A$, but $\Omega^{\prime \prime}[T]$ is disjoint with $A$, which is a contradiction.

Note also that for any $\tau \in T$ we have $a_{\tau} \in \mathscr{A}_{f}$. This follows because for each $\tau \in T$ we have $\left[T_{\tau}\right] \subseteq \Pi_{n} T_{\tau}(n)$ so

$$
\begin{aligned}
0 & <\mu\left(\prod_{n} T_{\tau}(n)\right)=\mu\left(\prod_{n}\left(f(n) \backslash a_{\tau}(n)\right)\right) \\
& =\lim _{n}\left(1-\left|a_{\tau}(0)\right| \cdot f(0)^{-1}\right)\left(1-\left|a_{\tau}(1)\right| \cdot f(1)^{-1}\right) \cdots\left(1-\left|a_{\tau}(n)\right| \cdot f(n)^{-1}\right)
\end{aligned}
$$

and therefore $\sum_{n} a_{\tau}(n) f(n)^{-1}<\infty$.
Now it is easy to convert the family $\left\{a_{\tau}: \tau \in T\right\}$ into a single $a$ satisfying our requirements.

I would like to thank J. Cichon for some helpful remarks.

1. Powers of transitive bases. We are going to prove the following results.

Theorem 1.1. $\Delta(\mathscr{L})=\Delta_{t}(\mathscr{L})$.
Theorem 1.2. $\Delta=\Delta_{t}(\mathscr{K})$.
Let us first deal with the measure case. We begin with the following simple lemma.
Lemma 1.3. If $A \in \mathscr{L}$, then there exists a sequence $\left\langle A_{n}\right\rangle_{n<\omega}$ and a function $l \in{ }^{\omega} \omega$ such that:
(a) $A \subseteq \cap_{m} \cup_{n>m} \cup A_{n}$,
(b) $\mu\left(\cup A_{n}\right)=2^{-n}$ for any $n<\omega$,
(c) for any $n<\omega$ the set $A_{n}$ is a finite subset of $\left\{\left[i 2^{-l(n)} ;(i+1) 2^{-l(n)}\right]: i \in \mathbf{Z}\right\}$.

We divide $\mathscr{L}$ into classes the following way: for $l \in{ }^{\omega} \omega$ let $\mathscr{L}(l)$ be the set of subsets $A$ of $\mathbf{R}$ for which there exists a sequence $\left\langle A_{n}\right\rangle_{n<\omega}$ with the properties (a), (b) and (c) of Lemma 1.3. Clearly $\mathscr{L}=\bigcup\left\{\mathscr{L}(l): l \in{ }^{\omega} \omega\right\}$ and it is easy to see that if $k \succ l$, then $\mathscr{L}(k) \supseteq \mathscr{L}(l)$. So if $D$ is a dominating family, then $\mathscr{L}=\bigcup\{\mathscr{L}(l)$ : $l \in D\}$. On the other hand if $D$ is not a dominating family, then it can be shown (cf. [M2]) that $\mathscr{L} \neq \bigcup\{\mathscr{L}(l): l \in D\}$. Thus $\Delta=\min \left\{|X|: X \subseteq{ }^{\omega} \omega \wedge \mathscr{L}=\bigcup\{\mathscr{L}(x)\right.$ : $x \in X\}\}$ and therefore $\Delta_{t}(\mathscr{L}) \geqslant \Delta$. To improve this estimation we have to work a little.

Let $l^{1}=\left\{x \in{ }^{\omega} \mathbf{R}:(\forall n x(n) \geqslant 0) \wedge\left(\sum_{n} x(n)<\infty\right)\right\}$. We call a set $X \subseteq l^{1}$ a dominating family if for any $y \in l^{1}$ there is $x \in X$ such that $\forall n^{\infty} y(n) \leqslant x(n)$. Let $\Delta\left(l^{1}\right)$ denote the minimal cardinality of a dominating family in $l^{1}$. Since for any $A \in \mathscr{L}$ there exists $x \in l^{1}$ such that $A \subseteq \cap_{m} \cup_{n>m}\left(q_{n}-x(n) ; q_{n}+x(n)\right)$, it follows that if $X$ is a dominating family in $l^{1}$, then $\left\{\bigcap_{m} \cup_{n>m}\left(q_{n}-x(n) ; q_{n}+x(n)\right)\right.$ : $x \in X\}$ is a base for $\mathscr{L}$. So $\Delta\left(l^{1}\right) \geqslant \Delta(\mathscr{L})$. Our strategy to prove Theorem 1.1 is to show that $\Delta_{t}(\mathscr{L}) \geqslant \Delta\left(l^{1}\right)$.

Lemma 1.4. $\Delta_{t}(\mathscr{L}) \geqslant \Delta\left(l^{1}\right)$.
Proof. We fix $f \in^{\omega}(\omega \backslash 1)$ such that $\sum_{n} f(n)^{-1}<\infty$. Note that for each $x \in l^{1}$ there is $g \in \Pi_{n} f(n) \cap \mathscr{A}_{f}$ such that $\forall n^{\infty} x(n) \leqslant g(n) f(n)^{-1}$. So

$$
\begin{array}{r}
\Delta\left(l^{1}\right)=\min \left\{|X|:\left(X \subseteq \mathscr{A}_{f} \cap \prod_{n} f(n)\right) \wedge\left(\forall y \in \mathscr{A}_{f} \cap \prod_{n} f(n)\right)\right. \\
\left.\left(\exists x \in X \forall n^{\infty} x(n) \geqslant y(n)\right)\right\} .
\end{array}
$$

Let $\mathscr{B}$ be a transitive base for $\mathscr{L}$. By Lemma 0.1 with each $B \in \mathscr{B}$ we can associate $a_{B} \in \mathscr{A}_{f}$ such that for any $b \in \mathscr{A}_{f}$ if $\Omega^{\prime \prime} b \subseteq B$, then $\forall n^{\infty} b(n) \subseteq a_{B}(n)$. For each $B \in \mathscr{B}$ we define a function $h_{B} \in \mathscr{A}_{f} \cap \prod_{n} f(n)$ : for any $n<\omega$ let $h_{B}(n)=\left|a_{B}(n)\right|$. We claim that for any $g \in \mathscr{A}_{f} \cap \prod_{n} f(n)$ there is $B \in \mathscr{B}$ such that $\forall n^{\infty} h_{B}(n) \geqslant g(n)$. Let $g \in \mathscr{A}_{f} \cap \Pi_{n} f(n)$ be fixed. For each $n \in \omega$ let $c(n)=\{0,1, \ldots, g(n)\}$. Then $\mathbf{Z}+\Omega^{\prime \prime} \bar{c} \in \mathscr{L}$. So there exists $B \in \mathscr{B}$ and $r \in \mathbf{R}$ such that $\mathbf{Z}+\Omega^{\prime \prime} \bar{c}+r \subseteq B$. We shall find $b \in \mathscr{A}_{f}$ such that $\Omega^{\prime \prime} \bar{b} \subseteq \mathbf{Z}+\Omega^{\prime \prime} \bar{c}+r$ and $|b(n)|=g(n)$ for any $n<\omega$. If this is done, then it follows by the definition of $h_{B}$ that $\forall n^{\infty} h_{B}(n) \geqslant g(n)$ and the proof of the lemma is finished.

Before defining $b$ we reveal the structure of sets $\Omega^{\prime \prime} \bar{a}$ for any $a \in \mathscr{A}_{f}$. For each $n<\omega$ let $F_{n}=(f(0) \cdots \cdot f(n))^{-1}$ and

$$
a^{*}(n)=\left[0, F_{n}\right]+a(n) \cdot F_{n}+\left\{0,1, \ldots,\left(F_{n-1}^{-1}-1\right)\right\} \cdot F_{n-1},
$$

where $F_{-1}=1$. Then $\Omega^{\prime \prime} \bar{a}=\bigcap_{m} \cup_{n>m} a^{*}(n)$. So for our $c$ we have $\mathbf{Z}+\Omega^{\prime \prime} \bar{c}+r=$ $\cap_{m} \cup_{n>m}\left(c^{*}(n)+\mathbf{Z}+r\right)$. For each $n<\omega$ we set $t_{n}=\operatorname{entier}\left(r \cdot F_{n}^{-1}\right)$ and $b(n)=$ $\left\{x-\operatorname{entier}\left(x F_{n-1}^{-1}\right): x \in\left\{t_{n}+1, \ldots, t_{n}+g(n)\right\}\right\}$. Then $b \in \mathscr{A}_{f}$ and for any $n<\omega$ we have

$$
\begin{aligned}
c^{*}(n)+\mathbf{Z}+r & =\left[0 ; F_{n}\right]+c(n) \cdot F_{n}+\mathbf{Z} \cdot F_{n-1}+r \\
& =\left[t_{n} F_{n} ;\left(t_{n}+g(n)+1\right) F_{n}\right]+\left(r-t_{n} F_{n}\right)+\mathbf{Z} F_{n-1} \\
& \supseteq\left[\left(t_{n}+1\right) F_{n} ;\left(t_{n}+g(n)+1\right) F_{n}\right]+\mathbf{Z} F_{n-1} \\
& =\left[0 ; F_{n}\right]+b(n) \cdot F_{n}+\mathbf{Z} F_{n-1} \\
& =b^{*}(n)+\mathbf{Z} .
\end{aligned}
$$

So

$$
\begin{aligned}
\Omega^{\prime \prime} \bar{b} & =\bigcap_{m} \bigcup_{n>m} b^{*}(n) \subseteq \bigcap_{m} \bigcup_{n>m}\left(b^{*}(n)+\mathbf{Z}\right) \\
& \subseteq \bigcap_{m} \bigcup_{n>m}\left(c^{*}(n)+\mathbf{Z}+r\right)=\mathbf{Z}+\Omega^{\prime \prime} \bar{c}+r
\end{aligned}
$$

We have defined $b$ with the required properties.
Note. It follows now that $\Delta\left(l^{1}\right)=\Delta(\mathscr{L})$. It has been first proved by Bartoszyński in [B].

Now we shall deal with the category case. It is much easier.
Proof of Theorem 1.2. The inequality $\Delta_{t}(\mathscr{K}) \geqslant \Delta$ follows from [M2] (see also Theorem 3.2), so we show only the opposite inequality. Let $D$ be a dominating family in ${ }^{\omega} \omega$ such that $D \subseteq{ }^{\omega}(\omega \backslash 1)$ and $|D|=\Delta$. For each $f \in D$ let

$$
G_{f}=\bigcap_{m} \bigcup_{n>m}\left(q_{n}-f(n)^{-1} ; q_{n}+f(n)^{-1}\right) .
$$

Then $G_{f}$ is a comeager set. We claim that the family $\left\{\mathbf{R} \backslash G_{f}: f \in D\right\}$ is a transitive base for $\mathscr{K}$.

Suppose that $A \in \mathscr{K}$. There exists a family $\left\{G_{n}: n<\omega\right\}$ of open dense sets such that $G=\bigcap_{n} G_{n} \subseteq \mathbf{R} \backslash A$. Since $G$ is comeager, $\cap\{G-q: q \in \mathbf{Q}\}$ is nonempty. Let $x \in \cap\{G-q: q \in \mathbf{Q}\}$. Then $\mathbf{Q} \subseteq G-x$, so for any $n<\omega$ we have $\mathbf{Q} \subseteq G_{n}-x$. Thus we can find a function $f_{n}$ such that

$$
\bigcup_{m}\left(q_{m}-f_{n}(m)^{-1} ; q_{m}+f_{n}(m)^{-1}\right) \subseteq G_{n}-x .
$$

Let $f \in D$ be such that for any $n<\omega$ we have $f \succ f_{n}$. Then $G_{f} \subseteq G_{n}-x$ for any $n<\omega$, so $G_{f} \subseteq G-x$ and $\left(\mathbf{R} \backslash G_{f}\right)+x \supseteq A$.

Note. (1) Let $\mathscr{N} \mathscr{W} \mathscr{D}$ be the ideal of nowhere dense subsets of $\mathbf{R}$. The above argument shows that $\Delta=\Delta_{t}(\mathscr{N} \mathscr{W} \mathscr{D})$.
(2) Let $\mathscr{I}$ be any ideal of subsets of a group $\mathbf{P}$. We can define an unbounded counterpart to $\Delta_{t}(\mathscr{I})$. Let $\lambda_{t}(\mathscr{I})=\min \{|\mathscr{A}|: \mathscr{A} \subseteq \mathscr{I} \wedge(\neg \exists I \in \mathscr{I})(\forall A \in \mathscr{A})$
$(\exists p \in \mathbf{P})(p \cdot A \subseteq I)\}$. By the method of this section we can prove that $\lambda_{t}(\mathscr{L})$ $=\operatorname{add}(\mathscr{L})$ and $\lambda_{t}(\mathscr{K})=\lambda$. Also interesting for its own right is that

$$
\lambda=\min \left\{|\mathscr{A}|: \mathscr{A} \subseteq \mathscr{L} \wedge\left(\neg \exists l \in{ }^{\omega} \omega\right)(\forall A \in \mathscr{A})(A \in \mathscr{L}(l))\right\} .
$$

2. Additivity. We construct a Cohen extension of the universe in which $\operatorname{add}_{t}(\mathscr{L})$ $>\operatorname{add}(\mathscr{L})$ and $\operatorname{add}(\mathscr{K})>\operatorname{add}(\mathscr{K})$ hold. It is convenient to give a set-theoretic equivalent for $\operatorname{add}_{t}(\mathscr{L})$. Let

$$
\begin{aligned}
& \lambda^{*}=\min \left\{|X|:\left(\exists f \in^{\omega}(\omega \backslash 1)\right)\left(X \subseteq \prod_{n} f(n) \wedge\left(\forall a \in \prod_{n}[f(n)]^{n}\right)\right.\right. \\
&\left.\left.(\exists x \in X)\left(\exists n^{\infty} x(n) \notin a(n)\right)\right)\right\} .
\end{aligned}
$$

We have the following easy but tedious lemma.
Lemma 2.1. Suppose that $D \subseteq{ }^{\omega}(\omega \backslash 1)$ is a dominating family. If a function $g \in{ }^{\omega} \omega$ is such that $\lim _{n} g(n)=\infty$, then

$$
\begin{aligned}
\lambda^{*}=\min \left\{|X|:(\exists f \in D)\left(X \subseteq \prod_{n} f(n) \wedge\right.\right. & \left(\forall a \in \prod_{n}[f(n)]^{\leqslant g(n)}\right) \\
& \left.\left.(\exists x \in X)\left(\exists n^{\infty} x(n) \notin a(n)\right)\right)\right\} .
\end{aligned}
$$

The following lemma justifies the introduction of $\lambda^{*}$.
Lemma 2.2. $\lambda^{*}=\operatorname{add}_{t}(\mathscr{L})$.
Proof. We first prove that $\lambda^{*} \leqslant \operatorname{add}_{t}(\mathscr{L})$. Let $A \in \mathscr{L}$ and $X \in[\mathbf{R}]^{<\lambda^{*}}$. We claim that $A+X \in \mathscr{L}$. Since $A+\mathbf{Z} \in \mathscr{L}$ we may assume that $A+\mathbf{Z}=A$. We fix a sequence $\left\langle A_{n}\right\rangle_{n<\omega}$ and a function $l \in{ }^{\omega} \omega$ obtained from Lemma 1.3 applied to $A$. Let $T_{n}$, for any $n<\omega$, be the set of such $K$, finite subsets of $\left\{\left[i 2^{-l(n)} ;(i+1) 2^{-l(n)}\right]\right.$ : $\left.i<2^{\prime(n)}\right\}$, that $\mu(\cup K)=2^{-n+1}$. Note that for any $x \in X$ and $n \in \omega$ there exists $K_{n}^{x} \in T_{n}$ such that $\cup A_{n}+x+\mathbf{Z} \subseteq \cup K_{n}^{x}+\mathbf{Z}$. Consider the family $\mathscr{X}=\left\{\left\langle K_{n}^{x}\right.\right.$ : $n \in \omega\rangle: x \in X\}$. Then $\mathscr{X} \subseteq \Pi_{n} T_{n}$ and $|\mathscr{X}|<\lambda^{*}$. So there exists $a \in \Pi_{n}\left[T_{n}\right]^{n}$ such that for any $x \in X$ we have $\forall n^{\infty} K_{n}^{x} \in a(n)$. Let $B_{n}=\bigcup \bigcup a(n)$ for any $n<\omega$. For any $x \in X$ we have $\forall n^{\infty} \cup K_{n}^{x} \subseteq B_{n}$, so $\forall n^{\infty} \cup A_{n}+x+\mathbf{Z} \subseteq B_{n}+\mathbf{Z}$. Therefore $A+X \subseteq \bigcap_{m} \bigcap_{n<m} B_{n}+\mathbf{Z}$, but $\bigcap_{m} \cup_{n>m} B_{n}$ is a null set because for any $n<\omega$ we have $\mu\left(B_{n}\right)<2^{-n+1} \cdot n$. Thus $A+X$ is a null set.

We now prove that $\lambda^{*} \geqslant \operatorname{add}_{t}(\mathscr{L})$. Suppose that $d \in{ }^{\omega} \omega$ is such that $d(n)>n$ for any $n \in \omega$. Let $X \subseteq \Pi_{n} 2^{d(n)}$ and $|X|<\operatorname{add}_{t}(\mathscr{L})$. We shall find $a \in \Pi_{n} \mathscr{P}\left(2^{d(n)}\right)$ such that $\forall n^{\infty}|a(n)| \leqslant 2^{n}$ and for any $x \in X$ we have $\forall n^{\infty} x(n) \in a(n)$. By Lemma 2.1 this suffices to prove the lemma.

We define functions $k, f \in{ }^{\omega} \omega$. For each $i<\omega$ let

$$
\begin{aligned}
k(i) & =n \text { iff } \sum_{m<n} 2^{d(m)-m} \leqslant i<\sum_{m \leqslant n} 2^{d(m)-m}, \\
f(i) & =2^{d(k(i))} .
\end{aligned}
$$

Note that $\sum_{i} f(i)^{-1}<\infty$. If $x \in X$, let $b_{x}(i)=\{x(k(i))\}$ for each $i<\omega$.

Claim. There exists $c \in \mathscr{A}_{f}$ such that for any $x \in X$ we have $\forall i^{\infty} b_{x}(i) \subseteq c(i)$.
To prove the claim it suffices, by Lemma 0.1, to show that $\cup\left\{\Omega^{\prime \prime} \bar{b}_{x}: x \in X\right\}$ is a null set. For each $i<\omega$, let $b(i)=\{0, f(i)-1\}$ and $F_{i}=(f(0) \cdot \cdots \cdot f(i))^{-1}$ (cf. Lemma 1.4). For $x \in X$ let $r_{x}=\sum_{i} x(k(i)) \cdot F_{i}$. Then for any $i>0$ and $x \in X$ we have

$$
\left[0 ; F_{i}\right]+b_{x}(i) \cdot F_{i}+\mathbf{Z} \cdot F_{i-1} \subseteq\left[0 ; F_{i}\right]+b(i) \cdot F_{i}+\mathbf{Z} \cdot F_{i-1}+r_{x}
$$

So for any $x \in X$ we have $\Omega^{\prime \prime} \bar{b}+\mathbf{Z}+r_{x} \supseteq \Omega^{\prime \prime} \bar{b}_{x}+\mathbf{Z}$. As $\Omega^{\prime \prime} \bar{b}+\mathbf{Z}$ is null and $|X|<\operatorname{add}_{t}(\mathscr{L})$, the set $\bigcup\left\{\Omega^{\prime \prime} b+\mathbf{Z}+r_{x}: x \in X\right\}$ is null. The claim is proved.

Now let $c$ be as in the claim and for each $n \in \omega$ let $a(n)=\cap\{c(i): k(i)=n\}$. Then for any $x \in X$ we have $\forall n^{\infty} x(n) \in a(n)$ and for any $n<\omega, a(n) \subseteq 2^{d(n)}$. It remains to show that $\forall n^{\infty}|a(n)| \leqslant 2^{n}$. This follows by

$$
\infty>\sum_{i}|c(i)| f(i)^{-1} \geqslant \sum_{n} 2^{d(n)-n} \cdot|a(n)| \cdot 2^{-d(n)}=\sum_{n}|a(n)| 2^{-n} .
$$

Note. From the above proof it follows that

$$
\operatorname{add}_{t}(\mathscr{L})=\min \left\{|X|:\left(\exists l \in{ }^{\omega} \omega\right)(X \in \mathscr{L}(l) \wedge \cup X \notin \mathscr{L})\right\} .
$$

It is proved in [M2] that $\operatorname{add}(\mathscr{L}) \leqslant \lambda$ and $\operatorname{add}(\mathscr{K}) \leqslant \lambda$. We have
Lemma 2.3. (a) $\operatorname{add}(\mathscr{L})=\min \left(\operatorname{add}_{t}(\mathscr{L}), \lambda\right)$.
(b) $\operatorname{add}(\mathscr{K})=\min \left(\operatorname{add}_{t}(\mathscr{K}), \lambda\right)$.

Proof. (a) Clearly $\operatorname{add}(\mathscr{L}) \leqslant \min \left(\operatorname{add}_{t}(\mathscr{L}), \lambda\right)$. So we prove the opposite inequality. Let $\mathscr{A} \subseteq \mathscr{L}$ be such that $|\mathscr{A}|<\min \left(\operatorname{add}_{t}(\mathscr{L}), \lambda\right)$. For each $A \in \mathscr{A}$ we fix a sequence $\left\langle A_{n}\right\rangle_{n<\omega}$ and a function $l^{A} \in{ }^{\omega} \omega$ obtained from Lemma 1.3. Since $|\mathscr{A}|<\lambda$ there exists a function $l \in{ }^{\omega} \omega$ such that for each $A \in \mathscr{A}$ we have $l^{A}<l$. So $\cup\left\{\mathscr{L}\left(l^{A}\right): A \in \mathscr{A}\right\} \subseteq \mathscr{L}(l)$ and therefore $\mathscr{A} \subseteq \mathscr{L}(l)$. By the note after Lemma 2.2 we get $\cup \mathscr{A} \in \mathscr{L}$.
(b) Again we only prove that $\operatorname{add}(\mathscr{K}) \geqslant \min \left(\operatorname{add}_{t}(\mathscr{K}), \lambda\right)$. Let $\mathscr{A} \subseteq \mathscr{K}$ be such that $|\mathscr{A}|<\min \left(\operatorname{add}_{t}(\mathscr{K}), \lambda\right)$. For each $A \in \mathscr{A}$ we fix a sequence $\left\langle G_{n}^{A}\right\rangle_{n<\omega}$ of dense open subsets of $\mathbf{R}$ such that $\cap_{n} G_{n}^{A} \subseteq \mathbf{R} \backslash A$. As in the proof of Theorem 1.2 we can find for each $A \in \mathscr{A}$ a real $x_{A}$ such that $\mathbf{Q}+x_{A} \subseteq \bigcap_{n} G_{n}^{A}$. So for each $A \in \mathscr{A}$ and each $n<\omega$ we can find a function $f_{n}^{A} \in{ }^{\omega} \omega$ such that

$$
\bigcup_{m}\left(q_{m}-f_{n}^{A}(m)^{-1} ; q_{m}+f_{n}^{A}(m)^{-1}\right) \subseteq G_{n}^{A}-x_{A} .
$$

Since $|\omega \times \mathscr{A}|<\lambda$ there exists a function $f \in{ }^{\omega} \omega$ such that $f \succ f_{n}^{A}$ for any $A \in \mathscr{A}$, $n<\omega$. It is clear that the comeager set

$$
G_{f}=\bigcap_{n} \bigcup_{m>n}\left(q_{m}-f(m)^{-1} ; q_{m}+f(m)^{-1}\right)
$$

is contained in $\bigcap_{n} G_{n}^{A}-x_{A}$ for any $A \in \mathscr{A}$. Since $|\mathscr{A}|<\operatorname{add}_{t}(\mathscr{K})$ it follows that the set $\cup\left\{\left(\mathbf{R} \backslash G_{f}\right)+x_{A}: A \in \mathscr{A}\right\}$ is meager. But it covers $\cup \mathscr{A}$.

Note. As in [B] we can prove that $\operatorname{add}_{t}(\mathscr{L}) \leqslant \operatorname{add}_{t}(\mathscr{K}) . \operatorname{So} \operatorname{add}(\mathscr{L})<\operatorname{add}(\mathscr{K})$ implies $\operatorname{add}_{t}(\mathscr{L})<\operatorname{add}_{t}(\mathscr{K})$. Therefore it is consistent with ZFC that add ${ }_{t}(\mathscr{L})$ $<\operatorname{add}_{t}(\mathscr{K})$.

We are ready to present our consistency result. The notation is standard (see [M1]). If $P$ is a notion of forcing, we call a subset $D \subseteq P$ predense if for any $p \in P$ there is $d \in D$ and $q \in P$ such that $q \leqslant p$ and $q \leqslant d$. A trivial notion of forcing is denoted by 1.

Theorem 2.4. If $Z F C$ is consistent, then the theory $Z F C+" \operatorname{add}_{t}(\mathscr{L})=\operatorname{add}_{t}(\mathscr{K})$ $=\omega_{2}+\lambda=\omega_{1}+2^{\omega}=\omega_{2} "$ is consistent .

Proof. For any $f \in{ }^{\omega} \omega$ we define a notion of forcing:

$$
\begin{gathered}
Q(f)=\left\{\langle n, a\rangle: n \in \omega \wedge\left(a \in \prod_{m}[f(m)]^{\leqslant m}\right) \wedge(\forall m>n|a(m)| \leqslant n)\right\}, \\
\left\langle n^{\prime}, a^{\prime}\right\rangle \leqslant\langle n, a\rangle \quad \text { iff }\left(n^{\prime} \geqslant n \wedge a\left|n=a^{\prime}\right| n \wedge \forall m a^{\prime}(m) \supseteq a(m)\right) .
\end{gathered}
$$

It is easy to see that $Q(f)$ satisifies ccc and that for any $A \subseteq Q(f)$ the property " $A$ is a maximal antichain in $Q(f)$ " is a $\Pi_{1}^{1}$ property of $A$ and $f$ (some reasonable coding by subsets of $\omega$ is presupposed).

The following property is crucial for our consideration: Let $G$ be a $Q(f)$ generic over some model $V$ of ZFC and let $a_{G}=\bigcup\{a:(\exists n)(\langle n, a\rangle \in G)\}$. Then $a_{G} \in$ $\Pi_{n}[f(n)]^{n}$ and in $V[G]$ we have $\left(\forall x \in V \cap \Pi_{n} f(n)\right)\left(\forall n^{\infty}\right)\left(x(n) \in a_{G}(n)\right)$.

Claim 1. If $Q(f) \Vdash(x \in \omega)$, then for each $k<\omega$ there is $n<\omega$ such that $(\forall\langle k, a\rangle \in Q(f))(\exists q \leqslant\langle k, a\rangle)(q \Vdash x<n)$.

Proof. Suppose not. Then there exists $k<\omega$ such that $(\forall n)(\exists\langle k, a\rangle \in$ $Q(f))(\langle k, a\rangle \Vdash x \geqslant n)$. Let $\left\langle a_{n}\right\rangle_{n<\omega}$ be a sequence such that for each $n<\omega$ we have $\left\langle k, a_{n}\right\rangle \Vdash(x \geqslant n)$. Then there exists $b$ such that $\langle k, b\rangle \in Q(f)$ and $(\forall m)\left(\exists n^{\infty}\right)\left(a_{n}|m=b| m\right)$. Let $\left\langle k^{\prime}, b^{\prime}\right\rangle$ be such that $\left\langle k^{\prime}, b^{\prime}\right\rangle \leqslant\langle k, b\rangle$ and $\left\langle k^{\prime}, b^{\prime}\right\rangle$ $\Vdash x=l$ for some $l<\omega$. Let $n>l$ be such that $a_{n}\left|2 k^{\prime}=b\right| 2 k^{\prime}$. Then $\left\langle 2 k^{\prime}, a_{n} \cup b^{\prime}\right\rangle$ $\in Q(f),\left\langle 2 k^{\prime}, a_{n} \cup b^{\prime}\right\rangle \Vdash x=l$ and $\left\langle 2 k^{\prime}, a_{n} \cup b^{\prime}\right\rangle \Vdash x \geqslant n$, which is a contradiction.

Suppose that $V \vDash \mathrm{ZFC}+\mathrm{GCH}$. Let $\left\langle f_{\xi}\right\rangle_{\xi<\omega_{2}}$ be an enumeration of sets hereditarily of power less than $\omega_{2}$ such that each set occurs $\omega_{2}$ times. We define iteration $\left\langle p_{\xi}\right\rangle_{\xi \leqslant \omega_{2}}$ :

$$
\begin{aligned}
& P_{0}=1, \quad P_{\xi}=\lim _{\eta<\xi} \operatorname{dir} \\
& P_{\eta} \quad \text { if } \xi \text { is limit, }, \\
& P_{\xi+1}= \begin{cases}P_{\xi} * Q\left(f_{\xi}\right) & \text { if } f_{\xi} \text { is a term appropriate for } P_{\xi} \text { and } P_{\xi} \Vdash f \in{ }^{\omega} \omega, \\
P_{\xi} * 1 & \text { if not. }\end{cases}
\end{aligned}
$$

It is clear that $P_{\omega_{2}}$ satisfies ccc and by standard forcing argument $P_{\omega_{2}} \Vdash 2^{\omega}=\omega_{2}$. To see that $P_{\omega_{2}} \Vdash \lambda^{*}=\omega_{2}$ let $G$ be $P_{\omega_{2}}$ generic over $V$, with $G_{\xi}=G \cap P_{\xi}$, $\xi<\omega_{2}$. Let $f \in{ }^{\omega} \omega \cap V[G]$ and $\left\{x_{\alpha}: \alpha<\kappa\right\} \subseteq V[G] \cap \prod_{n} f(n)$ with $\kappa<\omega_{2}$ given. By ccc there is $\zeta<\omega_{2}$ such that $f \in V\left[G_{\zeta}\right]$ and $\left\{x_{\alpha}: \alpha<\kappa\right\} \subseteq V\left[G_{\zeta}\right]$. Now a name for $f$ can be chosen to be of power hereditarily less than $\omega_{2}$, so as a name for $f$ we can take some $f_{\xi}$ for $\xi>\zeta$ and we can assume that $P_{\xi} \Vdash f_{\xi} \in{ }^{\omega} \omega$. Then $f \in V\left[G_{\xi}\right],\left\{x_{\alpha}\right.$ : $\alpha<\kappa\} \subseteq V\left[G_{\xi}\right]$ and by the crucial property in $V\left[G_{\xi+1}\right]$ we have $a \in \Pi_{n}[f(n)]^{n}$ such that $\forall n^{\infty} x_{\alpha}(n) \in a(n)$ for each $\alpha<\kappa$.

We shall show that $P_{\omega_{2}} \Vdash\left({ }^{\omega} \omega \cap V\right.$ is unbounded in $\left.{ }^{\omega} \omega\right)$, which will prove that $P_{\omega_{2}} \Vdash \lambda=\omega_{1}$. So let $f$ be a term such that $P_{\omega_{2}} \Vdash f \in{ }^{\omega} \omega$. By ccc we find countable
$\bar{P} \subseteq P_{\omega_{2}}$ such that for any $n \in \omega, p \in \bar{P}$ and $\xi \in \operatorname{supp}(p)$ the following sets are predense in $P_{\omega_{2}}$ :

$$
\begin{gathered}
\{q \in \bar{P}: \exists m \in \omega q \Vdash f(n)=m\}, \quad\left\{q \in \bar{P}: \exists m \in \omega q \Vdash f_{\xi}(n)=m\right\}, \\
\{q \in \bar{P}: \exists s \subseteq \omega \exists m \in \omega q \Vdash(\exists a p(\xi)=\langle m, a\rangle \wedge a(n)=s)\} .
\end{gathered}
$$

Let $S=\bigcup\{\operatorname{supp}(p): p \in \bar{P}\}$. We define the auxiliary iteration $\left\langle R_{\xi}\right\rangle_{\xi \leqslant \omega_{2}}$ :

$$
\begin{aligned}
& R_{0}=1, \quad R_{\xi}=\underset{\eta<\xi}{\lim \operatorname{dir}} R_{\eta}
\end{aligned} \quad \text { if } \xi \text { is limit, }, ~ \begin{array}{ll}
R_{\xi} *\left(Q\left(f_{\xi}\right)\right)^{V^{R_{\xi}}} & \text { if } \xi \in S, f_{\xi} \text { is a term appropriate for } R_{\xi}, \\
R_{\xi+1}= & \text { and } R_{\xi} \Vdash f_{\sigma} \in{ }^{\omega} \omega, \\
R_{\xi} * 1 & \text { if not. }
\end{array} .
$$

Claim 2. (a) $R_{\xi}$ can be canonically embedded into $P_{\xi}$ (we shall assume that $\left.R_{\xi} \subseteq P_{\xi}\right)$.
(b) $\{p \mid \xi: p \in \bar{P}\} \subseteq R_{\xi}$.
(c) Every dense subset of $R_{\xi}$ is predense in $P_{\xi}$ (this implies that if $\varphi$ is a sentence of the forcing language of $R_{\xi}$ and $R_{\xi} \Vdash \varphi$, then $P_{\xi} \Vdash V^{R_{\xi}} \vDash \varphi$ ).
(d) For $\xi \in S, f_{\xi}$ is a term appropriate for $R_{\xi}$ and $R_{\xi} \Vdash f_{\xi} \in{ }^{\omega} \omega$.

Proof (CF. [M1, §5]). The proof is by induction on $\xi$. The difficulty lies in proving (c) for the inductive step from $\xi$ to $\xi+1$ for $\xi \in S$. So suppose we have proved the claim for all $\eta \leqslant \xi$ and let $\xi \in S$. Let $D \subseteq R_{\xi+1}$ be dense. Let $E$ be a term appropriate for $R_{\xi}$ such that for any $p \in R_{\xi}$

$$
p \Vdash q \in E \quad \text { iff }\left(\forall p^{\prime} \leqslant p\right)\left(\exists\left\langle p^{\prime \prime}, q^{\prime \prime}\right\rangle \in D\right)\left(\exists p^{\prime \prime \prime} \leqslant p, p^{\prime \prime}\right)\left(p^{\prime \prime \prime} \Vdash q^{\prime \prime}=q\right)
$$

Then $R_{\xi} \Vdash\left(E\right.$ is a dense subset of $\left.Q\left(f_{\xi}\right)\right)$, so by the induction hypothesis

$$
P_{\xi} \Vdash V^{R_{\xi}} \vDash\left(E \text { is a dense subset of } Q\left(f_{\xi}\right)\right) .
$$

By $\Pi_{1}^{1}$ absoluteness between $V^{R_{\xi}}$ and $V^{P_{\xi}}$ it follows that $P_{\xi} \Vdash(E$ is a predense subset of $Q\left(f_{\xi}\right)$ ). Now it is routine to see that $D$ is predense in $P_{\xi+1}$. This ends our proof of Claim 2.

By Claim 2 it follows that $R_{\omega_{2}} \Vdash f \in{ }^{\omega} \omega$, and so it is enough to show that $R_{\omega_{2}} \Vdash\left(\exists g \in{ }^{\omega} \omega \cap V\right)\left(\exists n^{\infty}\right)(g(n) \geqslant f(n))$.

Let $T=\{t: t$ is a function from the finite subset of $S$ into $\omega\}$. For $t \in T$ let $K_{t}=\left\{p \in R_{\omega_{2}}: \operatorname{supp}(p)=\operatorname{dom}(t) \wedge(\forall \xi \in \operatorname{supp}(p))(p \Vdash \exists a p(\xi)=\langle t(\xi), a\rangle)\right\}$. Note that $\cup\left\{K_{t}: t \in T\right\}$ is dense in $R_{\omega_{2}}$. Using Claim 1 we can prove by induction on $|t|$ the following claim.

Claim 3. Suppose that $t \in T$ and $R_{\omega_{2}} \Vdash x \in \omega$. Then there exists $n<\omega$ such that for any $p \in K_{t}$ there is $q \leqslant p$ such that $q \Vdash x<n$.

Now let $\left\langle t_{i}\right\rangle_{i<\omega}$ be an enumeration with infinite repetitions of the set $T$. We define a function $g \in{ }^{\omega} \omega$ for each $m<\omega$ by

$$
\begin{gathered}
g(m)=\text { this } n \text { that Claim } 3 \text { holds with } t \text { replaced by } t_{m} \\
\text { and } x \text { replaced by } f(m) .
\end{gathered}
$$

Then $R_{\omega_{2}} \Vdash\left(\exists n^{\infty} g(n) \geqslant f(n)\right)$. To see this suppose that $p \in R_{\omega_{2}}, k<\omega$ and $p \Vdash(\forall n>k)(f(m)>g(m))$. Then there is $m>k$ and $p^{\prime} \in K_{t_{m}}$ such that $p^{\prime} \leqslant p$. By Claim 3 there is $q \leqslant p^{\prime}$ such that $q \Vdash f(m)<g(m)$, which is a contradiction.

Note. It is easy to see that $\lambda^{*} \leqslant \operatorname{add}(\mathscr{C})$, so if ZFC is consistent then ZFC + " $2^{\omega}$ $=\omega_{2}+\operatorname{add}(\mathscr{L})=\omega_{1}+\operatorname{add}(\mathscr{C})=\omega_{2} "$ is consistent.
3. Generalizations. We present some generalizations of theorems from $\S 1$.

Theorem 3.1. Let $\mathscr{K}(\mathbf{P})$ be the ideal of meager sets of a Polish group $\mathbf{P}$ (i.e. a topological group which is a Polish space without isolated points $)$. Then $\Delta_{t}(\mathscr{K}(\mathbf{P}))=\Delta$.

Proof. The proof of $\Delta_{t}(\mathscr{K}(\mathbf{P})) \leqslant \Delta$ goes along the same lines as that of Theorem 1.2. So we only show that $\Delta_{t}(\mathscr{K}(\mathbf{P})) \geqslant \Delta$.

Let $\delta$ be a complete metric in $\mathbf{P}$. We fix an increasing sequence $\left\langle E_{n}\right\rangle_{n<\omega}$ of finite subsets of $\mathbf{P}$, a sequence $\left\langle e_{n}\right\rangle_{n<\omega}$ of elements of $\mathbf{P}$ and sequences $\left\langle K_{n}\right\rangle_{n<\omega}$ and $\left\langle L_{n}\right\rangle_{n<\omega}$ of open neighbourhoods of the neutral element of $\mathbf{P}$ such that $U_{n} E_{n}$ is dense in $\mathbf{P}$ and for any $n<\omega$ the following conditions hold:
(a) $e_{n} \in E_{n+1} \backslash E_{n}^{-1} \cdot E_{n}$ and $E_{n} \cdot E_{n} \cup E_{n} \cdot e_{n} \subseteq E_{n+1}$,
(b) for any $x \in E_{n}$ we have $\delta\left(x e_{n}, x\right)<2^{-n}$,
(c) $L_{n} \cdot \bar{K}_{n+1} \subseteq K_{n}$ and $e_{n} \in L_{n} \backslash\left(K_{n+1} \cdot K_{n+1}^{-1} \cdot E_{n}^{-1} \cdot E_{n} \cdot K_{n+1} \cdot K_{n+1}^{-1}\right)$.

It is routine to see that this is possible. It is also easy to see that for any $n<\omega$ and $x, y \in P$ the set $x E_{n} K_{n+1}$ is disjoint with $y K_{n+1}$ or with $y e_{n} K_{n+1}$.

With each increasing function $g \in{ }^{\omega} \omega$ we associate an open dense set $H_{g}=$ $\cup_{n} E_{g(n)} K_{g(n)+1}$. And with each meager set $G \subseteq \mathbf{P}$ we associate a sequence $\left\langle G_{n}\right\rangle_{n<\omega}$ of open dense sets such that $\bigcap_{n} G_{n} \subseteq \mathbf{P} \backslash G$ and a function $h_{G} \in{ }^{\omega} \omega$ defined as follows: $h_{G}(0)$ is any element of $\omega$; if $h_{G}(n)$ is defined, then we set $h_{G}(n+1)$ to be the least possible $m$ such that there exists $b_{n} \in E_{m} \cap L_{h_{G}(n)}$ such that:
(i) $E_{h_{G}(n)} \cdot b_{n} \cdot K_{m} \subseteq G_{0} \cap \cdots \cap G_{h_{G}(n)}$,
(ii) $E_{h_{G}(n)} \cdot b_{n} \subseteq E_{m}$,
(iii) for any $x \in E_{h_{G}(n)}$ we have $\delta\left(x b_{n}, x\right)<2^{-h_{G}(n)}$.

We have the following claim from which it follows immediately that $\Delta_{t}(K(\mathbf{P})) \geqslant \Delta$.
Claim. If $\exists n^{\infty} g(n)>h_{G}(2 n)$, then for any $x \in \mathbf{P}$ we have $\mathbf{P} \backslash G \nsubseteq x H_{g}$.
Proof. We omit the subscript $G$. First note that $\left(\exists n^{\infty}\right)(h(n+1) \cap \operatorname{rng}(g) \subseteq$ $h(n)$ ). Let $x \in \mathbf{P}$. We define a sequence $\left\langle a_{m}\right\rangle_{m<\omega}$ of elements of $\mathbf{P}$ convergent to some $a \in(\mathbf{P} \backslash G) \backslash x H_{g}$. For $m \leqslant h(0)$ let $a_{m}$ be any element of $E_{m}$. If $n$ is such that $h(n+1) \cap \operatorname{rng}(g) \subseteq h(n)$ we set $a_{m}=a_{h(n)}$ for $h(n)<m<h(n+1)$ and $a_{h(n+1)}$ $=a_{h(n)} \cdot b_{n}$. If $n$ is such that $h(n+1) \cap \operatorname{rng}(g) \nsubseteq h(n)$ we set for $h(n)<m$ $\leqslant h(n+1): a_{m}=a_{m-1}$ if $x E_{m-1} K_{m}$ is disjoint with $a_{m-1} K_{m}$, or $a_{m}=a_{m-1} e_{m-1}$ if not. It is clear that $a_{m} \in E_{m}$ for any $m<\omega$, and that $\left\langle a_{m}\right\rangle_{m<\omega}$ is a Cauchy sequence in metric $\delta$. Let $a$ be its limit. It is not hard to see that $\forall n^{\infty}$ we have $a \in a_{h(n)} \cdot K_{h(n)}$ and $a \in a_{g(n)+1} \cdot K_{g(n)+1}$. By the definition of $h$, if for some $n, h(n+1) \cap \operatorname{rng}(g)$ $\subseteq h(n)$, then $a_{h(n+1)} K_{h(n+1)} \subseteq G_{0} \cap \cdots \cap G_{h(n)}$, so also $a \in G_{0} \cap \cdots \cap G_{h(n)}$. It follows that $\left(\exists n^{\infty}\right)\left(a \in G_{0} \cap \cdots \cap G_{n}\right)$ and consequently $a \in \bigcap_{n} G_{n} \subseteq \mathbf{P} \backslash G$. Now suppose, in order to obtain a contradiction, that $a \in x H_{g}$. Let $n$ be such that
$a \in x E_{g(n)} K_{g(n)+1}$. Then the set $x E_{g(n)} K_{g(n)+1}$ must be disjoint with $a_{g(n)+1} K_{g(n)+1}$. But this is impossible since $a$ is in the intersection.

The natural class of groups for which we would like to have Theorem 1.1 is the class of all Polish locally compact groups. We failed to prove this but we can prove the following.

Theorem 3.2. Let $\mathbf{P}$ be an abelian Polish locally compact group. Let $\mathscr{L}(\mathbf{P})$ be the ideal of Haar measure zero subsets of $\mathbf{P}$. Then $\Delta(\mathscr{L}(\mathbf{P}))=\Delta(\mathscr{L})$ and $\Delta_{t}(\mathscr{L}(\mathbf{P}))=$ $\Delta(\mathscr{L}(\mathbf{P}))$.

Proof. By a theorem of Sikorski [S, Theorem 32.5] there exists a Borel isomorphism $\varphi: \mathbf{R} \rightarrow \mathbf{P}$ such that for any set $X \subseteq \mathbf{R}$ we have $X \in \mathscr{L}$ iff $\varphi^{\prime \prime} X \in \mathscr{L}(\mathbf{P})$. So the first equality is clear.

The group $\mathbf{P}$ is isomorphic to $\mathbf{R}^{n} \times \mathbf{P}^{\prime}$, where $n<\omega$ and $\mathbf{P}^{\prime}$ has a compact subgroup $\mathbf{P}^{\prime \prime}$ such that $\mathbf{P}^{\prime} / \mathbf{P}^{\prime \prime}$ is discrete. If $n>0$, by the Fubini theorem we have $\Delta_{t}\left(\mathscr{L}\left(\mathbf{R}^{n} \times \mathbf{P}^{\prime}\right)\right) \geqslant \Delta_{t}(\mathscr{L}(\mathbf{R}))$, so $\Delta_{t}(\mathscr{L}(\mathbf{P})) \geqslant \Delta(\mathscr{L})$ and we are done. So suppose that $n=0$. It is not hard to see that $\Delta_{t}\left(\mathscr{L}\left(\mathbf{P}^{\prime}\right)\right) \geqslant \Delta_{t}\left(\mathscr{L}\left(\mathbf{P}^{\prime \prime}\right)\right)$. So we may confine ourselves with proving $\Delta_{t}\left(\mathscr{L}\left(\mathbf{P}^{\prime \prime}\right)\right) \geqslant \Delta(\mathscr{L})$.

The group $\mathbf{P}^{\prime \prime}$ being compact and separable is the inverse limit of a sequence $\left\langle\mathbf{T}^{n(i)} \times \mathbf{S}_{i}\right\rangle_{i<\omega^{\prime}}$ where $\mathbf{T}=\mathbf{R} / \mathbf{Z}$ and for any $i<\omega, n(i) \in \omega$ and $\mathbf{S}_{i}$ is a finite group. If $n(i)$ is not zero for some $i<\omega$, by the Fubini theorem and the form of the Haar measure in the inverse limits we obtain that $\Delta_{t}\left(\mathscr{L}\left(\mathbf{P}^{\prime \prime}\right)\right) \geqslant \Delta_{t}(\mathscr{L}(\mathbf{T}))$. But it is clear from the proof of Theorem 1.1 that $\Delta_{t}(\mathscr{L}(\mathbf{T}))=\Delta(\mathscr{L})$. So suppose that for any $i<\omega, n(i)=0$ and let $\mathbf{S}=\operatorname{liminv}_{i} \mathbf{S}_{i}, \mathbf{S}_{0}$ being trivial.

We define a function $f \in{ }^{\omega} \omega$ : for each $n>0$ let $f(n)=\left|\mathbf{S}_{n}\right| \cdot\left|\mathbf{S}_{n-1}\right|^{-1}, f(0)=1$. Without loss of generality we can assume that $\sum_{n} f(n)^{-1}<\infty$. For $i \leqslant j<\omega$ let $\varphi_{i j}$ : $\mathbf{S}_{j} \rightarrow \mathbf{S}_{i}$ be the mappings involved in taking the inverse limit of $\left\langle\mathbf{S}_{i}\right\rangle_{i<\omega}$. For each $n>0$ let $\mathbf{S}_{n}^{\prime}=\operatorname{ker} \varphi_{n-1, n}=\left\{s_{0}^{n}, \ldots, s_{f(n)-1}^{n}\right\}, s_{0}^{0}=$ the unique element of $\mathbf{S}_{0}$. Let $\boldsymbol{\vartheta}_{n-1, n}: \mathbf{S}_{n-1} \rightarrow \mathbf{S}_{n}$ in such a way that $\varphi_{n-1, n} \circ \boldsymbol{\vartheta}_{n-1, n}=\mathrm{id} \mid \mathbf{S}_{n-1}$. We define a mapping $\Xi: \Pi_{n} f(n) \rightarrow S$ : for $x \in \Pi_{n} f(n)$ let

$$
\Xi(x)(0)=s_{x(0)}^{0}, \quad \Xi(x)(n)=\vartheta_{n-1, n}(\Xi(x)(n-1))+s_{x(n)}^{n} .
$$

Then $\Xi$ is " $1-1$ " and onto. Let $\mu$ be the normalized Haar measure in $\mathbf{S}$ and $\nu$ be the canonical product measure in $\Pi_{n} f(n)$. For any $X \subseteq \Pi_{n} f(n)$ we have $\nu(X)=$ $\mu\left(\Xi^{\prime \prime} X\right)$. Now we can prove Lemma 0.1 with $S$ in place of $[0,1]$ and $\Xi$ in place of $\Omega$. Our strategy of proving $\Delta_{t}(\mathscr{L}(\mathbf{S})) \geqslant \Delta(\mathscr{L})$ is the same as in Theorem 1.1. Let $\mathscr{B}$ be a transitive base for $\mathscr{L}(\mathbf{S})$. For each $g \in \prod_{n} f(n) \cap \mathscr{A}_{f}$ we shall find $B \in \mathscr{B}$ and $h_{B} \in \mathscr{A}_{f} \cap \prod_{n} f(n)$ such that $\forall n^{\infty} h_{B}(n) \geqslant g(n)$. As in Theorem 1.1 this will suffice. Let $g$ be given. For each $n<\omega$ let $a(n)=\{g(0), g(1), \ldots, g(n-1)\}$. Then $\Xi " \bar{a} \in$ $\mathscr{L}(\mathbf{S})$. So there exists $B \in \mathscr{B}$ and $r \in S$ such that $\Xi{ }^{\prime \prime} \bar{a}+r \subseteq B$.

It is not hard to see that there exists $\Xi_{0}: \Pi_{n} f(n) \rightarrow S$ such that $\Xi_{0}^{\prime \prime} \bar{a}=\Xi^{\prime \prime} \bar{a}+r$. So by the modified Lemma 0.1 applied to $\Xi_{0}$ we have the canonical $c_{B} \in \mathscr{A}_{f}$ such that $\forall n^{\infty} b(n) \subseteq c_{B}(n)$. Let $h_{B}(n)=\left|c_{B}(n)\right|$ for $n<\omega$. Then $h_{B} \in \mathscr{A}_{f} \cap \Pi_{n} f(n)$ and $\forall n^{\infty} g(n) \leqslant h_{B}(n)$. One annoying detail remains: in defining $h_{B}$ we have used
$\Xi_{0}$. It can be overcome by noting that the definition of the $c_{B}$ 's for different $\Xi$ 's can be done so that the $h_{B}$ 's are the same.

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[^0]:    Received by the editors October 7, 1983.
    1980 Mathematics Subject Classification. Primary 03E35; Secondary 54A25.

