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INTEGER INVARIANTS OF CERTAIN EVEN-DIMENSIONAL KNOTS

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ABSTRACT. Integer invariants of certain simple $\mathbf{Z}$-torsion-free $2q$-knots, $q \geqslant 4$, are defined. It is shown that for $q \geqslant 5$, certain of these invariants must vanish, mod 2, if the knot is doubly-null-concordant.

Introduction. An *n-knot* is a locally flat piecewise-linear pair (S^{n+2}, S^n), both spheres being oriented. The exterior K of k is the closure of the complement of a regular neighbourhood of S^n. A $2q$-knot k is *simple* if K has the homotopy type of a circle below dimension q; or, equivalently, if $\pi_i(K) \cong \pi_i(S^1)$ for $1 \leqslant i < q$. Let $\tilde{K}$ be the universal cover of K; then duality theorems show that $H_q(\tilde{K})$ and $H_{q+1}(\tilde{K})$ are the only nontrivial homology groups of $\tilde{K}$ in such a case. If, in addition, $H_q(\tilde{K})$ has no $\mathbf{Z}$-torsion, then we refer to k as being $\mathbf{Z}$-torsion-free. The orientations of S^{2q} and S^{2q+2} yield a canonical generator t for $H_1(K) \cong H_1(S^1)$, via the Alexander and Poincaré duality isomorphisms, and the covering space action on $\tilde{K}$ makes $H_q(\tilde{K})$ and $H_{q+1}(\tilde{K})$ into Λ-modules, where $\Lambda = \mathbf{Z}[t, t^{-1}]$. Blanchfield duality yields a hermitian pairing into Λ_0/Λ, where Λ_0 is the field of fractions of Λ, which identifies $H_{q+1}(\tilde{K})$ as the conjugate dual of $H_q(\tilde{K})$. Conjugation in Λ is the linear extension of $t \to t^{-1}$.

For $q \geqslant 4$, the $\mathbf{Z}$-torsion-free simple $2q$-knots have been classified algebraically in [**1**] by their associated F-forms

$$\left(\mathscr{E}(\tilde{K}), H_q(\tilde{K}), p_q(\tilde{K}), [\ ,\ ]_{\tilde{K}}, \tau\langle\ ,\ \rangle_{\tilde{K}}\right),$$

where $p_q(\tilde{K})\colon H_q(\tilde{K}) \to H_q(\tilde{K})/2H_q(\tilde{K}) = \mathscr{H}_q(\tilde{K})$ is the quotient map, $\mathscr{H}_{q+1}(\tilde{K}) = H_{q+1}(\tilde{K})/2H_{q+1}(\tilde{K})$, $\Pi_{q+1}(\tilde{K}) = \pi_{q+1}(\tilde{K})/2\pi_{q+1}(\tilde{K})$, and

$$\mathscr{E}(\tilde{K})\colon \mathscr{H}_q(\tilde{K}) \overset{\omega}{\hookrightarrow} \Pi_{q+1}(\tilde{K}) \overset{h}{\twoheadrightarrow} \mathscr{H}_{q+1}(\tilde{K})$$

is a short exact sequence (s.e.s.) of Γ-modules ($\Gamma = \mathbf{Z}_2[t, t^{-1}]$).

The Blanchfield pairing induces the nonsingular hermitian pairing

$$^{\theta}\langle\ ,\ \rangle_{\tilde{K}}\colon \mathscr{H}_{q+1}(\tilde{K}) \times \mathscr{H}_q(\tilde{K}) \to \Gamma_0/\Gamma,$$

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which is related to the nonsingular hermitian pairing

$$[\ ,\ ]_{\tilde K}\colon \Pi_{q+1}(\tilde K) \times \Pi_{q+1}(\tilde K) \to \Gamma_0/\Gamma$$

by $[x, \omega(y)]_{\tilde K} = {}^{\theta}\langle h(x), y\rangle_{\tilde K}$ for all $x \in \Pi_{q+1}(\tilde K)$, $y \in \mathscr{H}_q(\tilde K)$.

To simplify notation, we shall identify $\mathscr{H}_{q+1}(\tilde K)$ with the conjugate dual $\mathscr{H}_q(\tilde K)^*$ and omit all mention of $\tilde K$ and q. Thus we have an s.e.s. of Γ-modules

$$\mathscr{H} \overset{\omega}{\rightarrowtail} \Pi \overset{h}{\twoheadrightarrow} \mathscr{H}^*$$

with pairings satisfying $[x, \omega(y)] = \langle h(x), y\rangle$.

1. The invariants. We shall be concerned with the special case of a Λ-module H which is the direct sum of Λ-modules $H_1, \ldots, H_n$; each H_i is annihilated by p_i^m, where $p_1, \ldots, p_n$ are distinct irreducible Laurent polynomials, with $p_i(t) = p_i(t^{-1})$. Moreover, we assume that, mod 2, each p_i is equal to the same polynomial p, also symmetric and irreducible. Then $\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_n$ in the obvious notation. Since Γ is a PID, $\mathscr{H}_i$ is a direct sum of Γ-modules $\mathscr{H}_{i,r}$ where $1 \leqslant r \leqslant m$ and each $\mathscr{H}_{i,r}$ is free over $\Gamma/(p^r)$. Similarly, $\mathscr{H}^* = \mathscr{H}_1^* \oplus \cdots \oplus \mathscr{H}_n^*$ and $\mathscr{H}_i^*$ is the direct sum of modules $\mathscr{H}_{i,r}^*$.

In a Γ-module M which is annihilated by p^m, let $\ker p^r = \{x \in M\colon p^r x = 0\}$. Set

$$M^r = \ker p^r/(\ker p^{r-1} + p \ker p^{r+1}).$$

Note that $\mathscr{H}^{*r} = \mathscr{H}_1^{*r} \oplus \cdots \oplus \mathscr{H}_n^{*r}$ and that $\mathscr{H}_i^{*r} = \mathscr{H}_{i,r}^*/p\mathscr{H}_{i,r}^*$. For $\ker p^r \subseteq \mathscr{H}^*$, let $\phi_r\colon \ker p^r \to \mathscr{H}^{*r}$ denote the quotient map. In Π, set $\Pi_r = h^{-1}(\ker p^r) \cap \ker p^{2r}$. From now on we shall assume that the map $\phi_r h|_{\Pi_r}\colon \Pi_r \to \mathscr{H}^{*r}$ is onto.

Let r be fixed, $1 \leqslant r \leqslant m$. Choose a basis for $\mathscr{H}_i^{*r}$, regarded as a vector space over the finite field $E = \Gamma/(p)$. Denote the union of these bases by $x_1, \ldots, x_N$, where it is understood that taking them in order gives first a basis for $\mathscr{H}_1^{*r}$, then a basis for $\mathscr{H}_2^{*r}$, and so on. For each j, choose $z_j \in \Pi_r$ so that $\phi_r h(z_j) = x_j$. Define $a_{ij} \in E$ by $[p^{2r-1}z_i, z_j] = b_{ij}/p$, $\psi(b_{ij}) = a_{ij}$, where $\psi\colon \Gamma \to E$ is the quotient map.

LEMMA. *a_{ij} is well defined.*

PROOF. The element z_j may be replaced by $z_j' = z_j + u_j + pv_j + \omega(w_j)$, where $h(u_j) \in \ker p^{r-1}$, $h(v_j) \in \ker p^{r+1}$, and $u_j + pv_j + \omega(w_j) \in \ker p^{2r}$. Note that $[p^{2r-1}z_i', pv_j] = [p^{2r}z_i', v_j] = 0$, and so

$$\begin{aligned}
[p^{2r-1}z_i', z_j'] &= [p^{2r-1}z_i', z_j + u_j + \omega(w_j)]\\
&= [p^{2r-1}z_i', z_j + u_j] + \langle p^{2r-1}h(z_i'), w_j\rangle\\
&= [p^{2r-1}z_i', z_j + u_j] \text{ since } p^r h(z_i') = 0\\
&= [p^{2r-1}(z_i + u_i + pv_i), z_j + u_j] + \langle w_i, p^{2r-1}h(z_j + u_j)\rangle\\
&= [p^{2r-1}(z_i + u_i + pv_i), z_j + u_j].
\end{aligned}$$

Noting that $p^{r-1}u_i = \omega(\alpha_i)$, since $p^{r-1}h(u_i) = 0$, and that $[p^{2r}v_i, z_j] = [v_i, p^{2r}z_j] = [v_i, 0] = 0$, we have

$$\begin{aligned}\left[p^{2r-1}z_i', z_j\right] &= \left[p^{2r-1}(z_i + u_i + pv_i), u_j\right] + \left[p^{2r-1}z_i, z_j\right] + \left[p^{2r-1}u_i, z_i\right]\\ &= \langle p^r h(z_i + u_i + pv_i), \alpha_j\rangle + \left[p^{2r-1}z_i, z_j\right] + \langle p^r h(z_j), \alpha_i\rangle\\ &= \left[p^{2r-1}z_i, z_j\right],\end{aligned}$$

since $h(z_i + u_i + pv_i), h(z_i) \in \ker p^r$. $\square$

Note that the matrix $A = (a_{ij})$ is hermitian. If we make a different choice of basis for $\mathscr{H}_i^{*r}$, then we obtain a transformation matrix B which is a block diagonal matrix, and A is replaced by BAB^*.

Write A in block form as (A_{ij}), where i and j run from 1 to n, corresponding to the choice of bases above. Then BAB^* has the same form, A_{ij} being replaced by $B_i A_{ij} B_j^*$, where $B = \operatorname{diag}(B_1, \ldots, B_n)$ and each B_i is nonsingular. Set $\sigma_{ij} = \operatorname{rank}(A_{ij})$; then these do not depend on the choice of bases, and so are invariants of the knot k. Note also that $\sigma_{ij}^r = \sigma_{ji}^r$, because A is hermitian.

2. Doubly-null-concordant knots. An n-knot is doubly-knot-concordant if it is a cross section of the trivial $(n+1)$-knot. For a simple $\mathbf{Z}$-torsion-free $2q$-knot k, $q \geqslant 5$, to be doubly-null-concordant, it is necessary and sufficient that its F-form have the following structure [2]:

(i) $H = H_+ \oplus H_-$.

(ii)

$$\mathscr{E} = \mathscr{E}_+ \oplus \mathscr{E}_-, \text{ where } \mathscr{E}_- = \mathscr{E}_+^*, \quad \text{and}$$

$$\mathscr{E}_+ : \mathscr{H}_+ \overset{\omega}{\rightarrowtail} \Pi \overset{h}{\twoheadrightarrow} \mathscr{H}_-^*, \quad \mathscr{E}_- : \mathscr{H}_- \overset{h^*}{\rightarrowtail} \Pi^* \overset{\omega^*}{\twoheadrightarrow} \mathscr{H}_+^*.$$

(iii) Π and Π^* are self-annihilating under [,]; indeed, [,] is given by the evaluation map $\Pi^* \times \Pi \to \Gamma_0/\Gamma$.

Note that, in such a case, $\Pi_{q+1}(\tilde{K}) = \Pi \oplus \Pi^*$, and the ω, h of the Introduction are $\omega \oplus h^*$, $h \oplus \omega^*$, respectively.

THEOREM. *Let k be a doubly-null-concordant $2q$-knot, $q \geqslant 5$, which satisfies the hypotheses of* §1. *Then σ_{ii}^r is even, for all i and r.*

PROOF. We know that H splits as $H_1 \oplus \cdots \oplus H_n$ and as $H_+ \oplus H_-$, in both cases as a Λ-module. For each i, $H_i = \{x \in H: p_i^m x = 0\} = \ker p_i^m$. If $x_\varepsilon \in H_\varepsilon$, then $p_i^m(x_+ + x_-) = 0$ if and only if $p_i^m x_+ = 0$ and $p_i^m x_- = 0$. Thus $H_i = H_{i,+} \oplus H_{i,-}$, where $H_{i,\varepsilon} = H_i \cap H_\varepsilon$, $\varepsilon = \pm$, and so $\mathscr{H}_\varepsilon = \mathscr{H}_{1,\varepsilon} \oplus \cdots \oplus \mathscr{H}_{n,\varepsilon}$ in the obvious notation. Moreover, $\mathscr{H}_\varepsilon^r = \mathscr{H}_{1,\varepsilon}^r \oplus \cdots \oplus \mathscr{H}_{n,\varepsilon}^r$ for each r.

When choosing the basis $x_1, \ldots, x_N$ of $\mathscr{H}^{*r}$, we can therefore arrange for this to be the union of bases for $\mathscr{H}_{1,+}^{*r}, \mathscr{H}_{1,-}^{*r}, \mathscr{H}_{2,+}^{*r}, \mathscr{H}_{2,-}^{*r}, \ldots, \mathscr{H}_{n,+}^{*r}, \mathscr{H}_{n,-}^{*r}$ in that order. Since Π and Π^* are self-annihilating under the pairing, each diagonal block A_{ii} of the hermitian matrix A has the form

$$\begin{pmatrix} 0 & \beta_i \\ \beta_i^* & 0 \end{pmatrix},$$

and so $\sigma_{ii}^r = \operatorname{rank} \beta_i + \operatorname{rank} \beta_i^* = 2 \operatorname{rank} \beta_i$ is even.

3. Examples. The first two examples are given in [**1**, p. 52]. Let $p_1(t) = t^{-1} - 1 + t$, $p_2(t) = t^{-1} - 3 + t$, and set $H = \Lambda/(p_1) \oplus \Lambda/(p_2)$ with generators x_1, x_2 for $\mathscr{H}^*$. Let $\Pi = \Gamma/(p^2) \oplus \Gamma/(p^2)$ with generators z_1, z_2 so that $h(z_i) = x_i$ for $i = 1, 2$. Define hermitian forms $[\ ,\ ]$ and $\langle\ ,\ \rangle$ on Π by

(i) $[z_1, z_1] = 1/p^2 = [z_2, z_2]$, $[z_1, z_2] = 0$,

(ii) $\langle z_1, z_1 \rangle = 0 = \langle z_2, z_2 \rangle$, $\langle z_1, z_2 \rangle = 1/p^2$.

In the first case, the matrix A is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and in the second case it is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The invariants σ_{ij}^r are

(i) $\sigma_{11}^1 = \sigma_{22}^1 = 1$, $\sigma_{12}^1 = 0$;

(ii) $\sigma_{11}^1 = \sigma_{22}^1 = 0$, $\sigma_{12}^1 = 1$.

Note that the first F-form belongs to a knot which is not doubly-null-concordant; taking $\mathscr{H}_+^* = \langle x_1 \rangle$, $\mathscr{H}_-^* = \langle x_2 \rangle$, it is easy to see that the second F-form represents a doubly-null-concordant knot.

Defining the hermitian form in the obvious way, we can create F-forms giving rise to a matrix A which is

(iii) $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and

(iv) $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Neither of the corresponding knots is doubly-null-concordant, by the Theorem.

References

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