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SHORTER NOTES

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A SIMPLE INTUITIVE PROOF OF A THEOREM IN DEGREE THEORY FOR GRADIENT MAPPINGS

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Dedicated to Béla

ABSTRACT. We give a simple, intuitive proof of a known theorem: the degree of the gradient of a coercive functional on a large ball in $\mathbf{R}^n$ is one.

Introduction. This theorem was originally proved by Krasnosel'skii [**1968**]. His method of proof involves the following of gradient lines. Nirenberg [**1981**] has mentioned in his survey that the result seems intuitively clear but he knows no elementary proof of it. We provide a proof which is simple and brings to light the essential reason why the theorem is true. We make use of the Poincaré-Hopf Theorem and the fact that the degree of the gradient at a local maximum is $(-1)^n$. This last fact is originally due to Rothe [**1950–51**] but Rabinowitz [**1975**] has provided a simple proof using the Poincaré-Hopf Theorem.

Recently, Amann [**1982**] has given a proof for the Hilbert space case modeled on the ideas of Krasnosel'skii. Our proof does not extend readily. Nevertheless, it is hoped that such an extension can be found.

Theorem and proof.

THEOREM. *Let $\phi \in C^1(\mathbf{R}^n, \mathbf{R})$ be coercive. That is, $\phi \uparrow +\infty$ as $|y| \to \infty$. Suppose that $\nabla\phi \neq 0$ outside a ball B about the origin. Then, for any ball B' which properly contains B,*

$$\deg(\nabla\phi, B', 0) = 1.$$

HEURISTIC OUTLINE OF PROOF. The fact that the functional is coercive means that if we stereographically project $\mathbf{R}^n \cup \{\infty\}$ onto S^n, sending $\{\infty\}$ to the north pole, the functional can be considered as a functional on S^n with a local maximum (with value ∞) at the north pole. If this functional is smooth, the total degree of the gradient is $1 + (-1)^n$ by the Poincaré-Hopf Theorem. Since the local degree of the gradient at a local maximum is $(-1)^n$, the excision principle for degree implies that the degree of the gradient, omitting a ball about the north pole, is one.

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To do this correctly we will need two lemmas. ϕ is bounded below and we suppose that this lower bound is 1. Then $\psi = -1/\phi$ is bounded below by -1, above by zero, and converges to zero as $|y| \to \infty$.

LEMMA 1. $\deg(\nabla\phi, B', 0) = \deg(\nabla\psi, B', 0)$.

PROOF. The homotopy

$$t\nabla\phi + (1-t)\nabla\psi = (t + (1-t)/\phi^2)\nabla\phi, \qquad t \in [0,1],$$

does not vanish on $\partial B'$. The result follows.

By stereographic projection we may consider ψ to be a continuous function on S^n where $\{\infty\}$ is located at the north pole and $\tilde{B}'$ denotes the image of B'.

Observe that $\nabla\psi$, as defined on S^n, is a section of TS^n. ψ may not be differentiable at $\{\infty\}$. Therefore, we need the following

LEMMA 2. *There exists a function $U \in C^1(S^n, \mathbf{R})$ which is homotopically equivalent to ψ on $\tilde{B}'$, has the same critical points as ψ, and has an isolated maximum at $\{\infty\}$.*

PROOF. Consider a coordinate patch on S^n with origin at $\{\infty\}$, with coordinates x such that $|x| = R$ runs between 0 and 1, and where $|x| = R = 1$ denotes the south pole.

By our hypothesis,

$$\begin{aligned} |\psi| &\le f(R), \qquad & |x| \le R, \\ |\nabla\psi| &\le K(R), \qquad & |x| \ge R, \end{aligned}$$

where f is differentiable on $(0,1)$ and monotone increasing, and K is continuous and monotone decreasing. f can be chosen to be strictly monotone so that f^{-1} exists and is differentiable.

Let $\xi \le 0$. Define

$$U'(\xi) = -\xi/(K(f^{-1}(\xi))). \tag{1}$$

U' is strictly positive when $\xi \ne 0$ and is monotonically decreasing. Then, $U(\xi) = \int_0^\xi U'(\tau)\,d\tau$ can be used to define a new functional on S^n by composition: $U \equiv U(\psi(x))$.

On $|x| = R, |\nabla U| = |U'(\psi(x))\nabla\psi| \le |U'(-f(R))\nabla\psi|$, since $\psi \ge -f(R)$ and U' is monotone. By (1),

$$U'(-f(R)) = \frac{f(R)}{K(f^{-1}(f(R)))} = \frac{f(R)}{K(R)}.$$

Thus $|\nabla U| \le (f(R)/K(R)) \cdot K(R) = f(R)$, which tends to zero as x does. Since $\nabla U = U(\psi(x))\nabla\psi$, $\nabla U = 0$ only when $\nabla\psi = 0$ or when $x = 0$. The homotopy

$$t\nabla\psi + (1-t)\nabla U = (t + (1-t)U')\nabla\psi, \qquad t \in [0,1],$$

does not vanish on B'. Therefore,

$$\deg(\nabla\psi, \tilde{B}', 0) = \deg(\nabla U, \tilde{B}', 0).$$

PROOF OF THEOREM. We have $U \in C^1(S^n, \mathbf{R})$. ∇U is a continuous section of TS^n. By the Poincaré-Hopf Theorem,

$$\deg(\nabla U, S^n, 0) = 1 + (-1)^n.$$

Since $\nabla U \neq 0$ in $S^n - \overline{\tilde{B}'}$ except at $\{\infty\}$, where U has an isolated local maximum, $\deg(\nabla U, S^n - \overline{\tilde{B}'}, 0) = (-1)^n$. See Rabinowitz [**1975**]. By the excision principle for degree,

$$\deg(\nabla U, S^n, 0) = \deg(\nabla U, \tilde{B}', 0) + \deg(\nabla U, S^n - \overline{\tilde{B}'}, 0).$$

Therefore, $1 + (-1)^n = \deg(\nabla U, \tilde{B}', 0) + (-1)^n$ implies that $\deg(\nabla U, \tilde{B}', 0) = 1$. By Lemmas 1 and 2,

$$\begin{aligned} \deg(\nabla \phi, B', 0) &= \deg(\nabla \psi, B', 0) = \deg(\nabla \psi, \tilde{B}', 0) \\ &= \deg(\nabla U, \tilde{B}', 0) = 1. \end{aligned}$$

The proof is finished.

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