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COMMUTATIVE MONOID RINGS AS HILBERT RINGS

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ABSTRACT. Let S be a cancellative monoid with quotient group of torsion-free rank α. We show that the monoid ring $R[S]$ is a Hilbert ring if and only if the polynomial ring $R[\{X_i\}_{i\in I}]$ is a Hilbert ring, where $|I| = \alpha$.

Assume that R is a commutative unitary ring and G is an abelian group. The first research problem listed in [**K**, Chapter 7] is that of determining equivalent conditions in order that the group ring $R[G]$ should be a Hilbert ring. Matsuda has considered this question in [**M1**, §6]. His results show that for G finitely generated, $R[G]$ and R are simultaneously Hilbert rings; if G is not finitely generated, then $R[G]$ a Hilbert ring implies that R is a Hilbert ring, but the converse fails. In [**M2**, §5] Matsuda considers briefly the corresponding Hilbert-ring-characterization problem for a monoid ring $R[S]$, where S is torsion-free and cancellative. Let α be the torsion-free rank of G and let $\{X_i\}_{i\in I}$ be a set of indeterminates over R of cardinality α. In Corollary 1 we show that $R[G]$ and $R[\{X_i\}_{i\in I}]$ are simultaneously Hilbert rings. While the problem of determining equivalent conditions for $R[\{X_i\}_{i\in I}]$ to be a Hilbert ring has not been completely resolved, it has been worked on extensively [**Kr**, **Go**, **L**, **G1**, **H**], and a significant body of positive results exists concerning this problem. The proof of Theorem 1 suggests the following conjecture: if S is a cancellative commutative monoid with quotient group G, then $R[S]$ is a Hilbert ring if and only if $R[G]$ is a Hilbert ring; this result is established in Theorem 2.

All monoids considered are assumed to be commutative, and rings are assumed to be commutative and unitary. The statement of Theorem 1 uses the following terminology. For a cardinal number α, an extension ring T of R is said to be *α-generated over R* if T is generated over R by a set of cardinality at most α.

THEOREM 1. *Assume that $X = \{X_i\}$ is a set of indeterminates of cardinality α over the ring R. Denote by Z^α the direct sum of α copies of the additive group Z of integers. The following conditions are equivalent.*

(1) *$R[X]$ is not a Hilbert ring.*

(2) *There exists a prime ideal P of R such that R/P admits an α-generated extension ring that is a G-domain, but not a field.*

(3) *The group ring $R[Z^\alpha]$ is not a Hilbert ring.*

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PROOF. (1) $\Rightarrow$ (2). If $R[X]$ is not a Hilbert ring, then there exists a G-ideal Q of $R[X]$ that is not maximal [**G2**, Theorem 31.8]. Hence, if $P = Q \cap R$, then $R[X]/Q$ is an α-generated extension ring of R/P that is a G-domain, but not a field.

(2) $\Rightarrow$ (3). Let $D = R/P$ and let $J = D[\{a_i\}_{i \in I}]$ be an α-generated extension domain of D that is a G-domain, but not a field. With an eye toward proving Theorem 2, we show that if T is a subring of $D[\{X_i\}, \{X_i^{-1}\}] \simeq D[Z^\alpha]$ containing $D[X]$, then T is not a Hilbert ring. Since J is a G-domain, it has nonzero pseudoradical [**G3**, Theorem 3; **G2**, Theorem 31.1; **Ka**, §§1–3]. Choose a nonzero element b in the pseudoradical of J. Then $1 + ba_i$ is a unit of J for each i, so

$$D\left[\{1 + ba_i\}, \left\{(1 + ba_i)^{-1}\right\}\right] \subseteq J.$$

The D-homomorphism of $D[\{X_i\}]$ onto $D[\{1 + ba_i\}]$ determined by $X_i \to 1 + ba_i$ admits an extension σ to a D-homomorphism of $D[Z^\alpha]$ onto

$$D\left[\{1 + ba_i\}, \left\{(1 + ba_i)^{-1}\right\}\right],$$

and under σ we have

$$J \supseteq \sigma(T) \supseteq D\left[\{1 + ba_i\}\right] = D\left[\{ba_i\}\right].$$

Now

$$\sigma(T)[b^{-1}] \supseteq D\left[\{ba_i\}, b^{-1}\right] \supseteq D\left[\{a_i\}, b^{-1}\right] \supseteq J[b^{-1}],$$

the quotient field of J. Consequently, $\sigma(T)[b^{-1}]$ is the quotient field of J, and $\sigma(T)$ is a G-domain. If $\sigma(T)$ were a field, then $\sigma(T)[b^{-1}]$, and hence J, would be algebraic over $\sigma(T)$, and this contradicts the fact that J is not a field. Therefore $\sigma(T)$, and hence T, is not a Hilbert ring. In particular, $D[Z^\alpha] \simeq R[Z^\alpha]/P[Z^\alpha]$ is not a Hilbert ring, and hence neither is $R[Z^\alpha]$.

(3) $\Rightarrow$ (1). If α is finite, then (3) implies that R is not a Hilbert ring, and hence neither is $R[X]$. If α is infinite, then $R[Z^\alpha]$ is α-generated over R, and thus is a homomorphic image of $R[X]$. Therefore (1) follows in this case as well. This completes the proof of Theorem 1.

COROLLARY 1. *Assume that G is an abelian group of torsion-free rank α. If $\{X_i\}_{i \in I}$ is a set of indeterminates over R of cardinality α, then $R[G]$ and $R[\{X_i\}_{i \in I}]$ are simultaneously Hilbert rings. In particular, if α is finite, then $R[G]$ is a Hilbert ring if and only if R is a Hilbert ring.*

PROOF. Choose a free subgroup F of G such that G/F is a torsion group. Then $R[G]$ is integral over $R[F]$, and hence $R[G]$ and $R[F]$ are simultaneously Hilbert rings. Since $F \simeq Z^\alpha$, the result then follows from Theorem 1.

If Z_0 denotes the additive monoid of nonnegative integers, then $R[\{X_i\}_{i \in I}]$ is isomorphic to the monoid ring of Z_0^α over R, where $\alpha = |I|$. Moreover, Z^α is the quotient group of Z_0^α. Does the equivalence of conditions (1) and (3) of Theorem 1 generalize to the case of an arbitrary cancellative monoid and its quotient group? Theorem 2 shows that this question has an affirmative answer. Theorem 1 is used in the proof of Theorem 2.

THEOREM 2. *If S is a cancellative monoid with quotient group G, then $R[S]$ and $R[G]$ are simultaneously Hilbert rings.*

PROOF. Let α be the torsion-free rank of G (according to the terminology of [**G4**, p. 165], α is also referred to as the torsion-free rank of S). If α is finite, then Theorem 2 follows from [**G5**, Corollary 1]. If α is infinite, then choose a free subset F of S such that G/H is a torsion group, where H is the subgroup of G generated by F. Let $T = S \cap H$. We observe that $R[S]$ is integral over $R[T]$. This is true since, for $s \in S$, there exists a positive integer n such that $ns \in H \cap S = T$. Since the extensions $R[H] \subseteq R[G]$ and $R[T] \subseteq R[S]$ are integral, $R[G]$ and $R[H]$ are simultaneously Hilbert rings, and the same is true of $R[T]$ and $R[S]$. This reduces the proof of Theorem 2 to the case where $G = Z^\alpha$ and S is a submonoid of G containing Z_0^α.

Suppose $R[Z^\alpha]$ is not a Hilbert ring. There exist a prime ideal P of R and an α-generated extension of R/P which is G-domain but not a field. The proof that (2) implies (3) in Theorem 1 then shows that no domain between $(R/P)[Z_0^\alpha]$ and $(R/P)[Z^\alpha]$ is a Hilbert ring. In particular, $(R/P)[S]$ is not a Hilbert ring, so neither is $R[S]$. Conversely, if $R[S]$ fails to be a Hilbert ring then, since α is infinite, $|S| = \alpha$ and $R[S]$ is a homomorphic image of $R[Z_0^\alpha]$; therefore neither $R[Z_0^\alpha]$ nor $R[Z^\alpha]$ is a Hilbert ring in this case.

We referred in the introduction to work that has been done on the problem of determining conditions under which a polynomial ring in infinitely many indeterminates over a Hilbert ring is again a Hilbert ring. To illustrate the relation between some of this work and Theorems 1 and 2, we record a result labelled as Theorem 3. Part (a) of this result, a restatement of [**G5**, Corollary 1], follows from Theorems 1 and 2; part (b) uses the same two theorems and Theorem 2.9 of [**G1**], while part (c), which generalizes (b), is a consequence of Theorems 1 and 2 and [**H**, Theorem 1].

THEOREM 3. *Assume that R is a commutative unitary ring and that S is a cancellative monoid of torsion-free rank α.*

(a) *If α is finite, then $R[S]$ is a Hilbert ring if and only if R is a Hilbert ring.*

Suppose that α is infinite.

(b) *If R is a field, then $R[S]$ is a Hilbert ring if and only if $\alpha < |R|$.*

(c) *If R is a Noetherian ring (or, more generally, if* Spec(R) *is Noetherian and R satisfies d.c.c. on prime ideals), then $R[S]$ is a Hilbert ring if and only if the following conditions are satisfied.*

(i) $|R/M| > \alpha$ *for each maximal ideal M of R.*

(ii) *For each nonmaximal prime P of R, the set of primes Q of R, such that $Q > P$ and* $\mathrm{ht}(Q/P) = 1$, *has cardinality greater than α.*

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