

NOTE ON THE TRIVIAL ZEROS OF DIRICHLET L -FUNCTIONS

TOM M. APOSTOL

ABSTRACT. The trivial zeros of Dirichlet L -functions are located without use of the functional equation.

If χ is a Dirichlet character mod k , then the Dirichlet L -function $L(s, \chi)$ has trivial zeros whose location is determined by the number $\chi(-1)$ which is ± 1 . If $\chi(-1) = 1$ there are zeros at the points $s = 0, -2, -4, -6, \dots$, and if $\chi(-1) = -1$ there are zeros at the points $s = -1, -3, -5, -7, \dots$. In other words, for every integer $n \geq 0$,

$$(1) \quad \chi(-1) = (-1)^n \text{ implies } L(-n, \chi) = 0.$$

This result is usually derived from the functional equation for the L -functions which is valid only for primitive characters. A separate argument is then needed to treat the nonprimitive characters (see [2, pp. 215–216]).

This note gives a short proof of (1) which does not require the functional equation for L -functions and which applies to all Dirichlet characters, primitive or not. It makes use of the representation [1, p. 255]

$$(2) \quad L(s, \chi) = k^{-s} \sum_{r=1}^{k-1} \chi(r) \zeta(s, r/k)$$

which holds for all complex s and all Dirichlet characters χ mod k . Here $\zeta(s, a)$ is the Hurwitz zeta function defined for $R(s) > 1$ by the series

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s},$$

where $0 < a \leq 1$; it can be extended as a meromorphic function to the entire s -plane by the contour integral

$$(3) \quad \zeta(s, a) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1} e^{az}}{1 - e^z} dz,$$

where C is a loop around the negative real axis [1, p. 253]. The only property of $\zeta(s, a)$ we need to prove (1) is the formula

$$(4) \quad \zeta(-n, 1-a) = (-1)^{n+1} \zeta(-n, a)$$

Received by the editors January 19, 1984.

1980 *Mathematics Subject Classification*. Primary 10H08.

Key words and phrases. Dirichlet L -function, trivial zeros.

©1985 American Mathematical Society
 0002-9939/85 \$1.00 + \$.25 per page

for integer n . This follows at once by replacing z by $-z$ in the contour integral (3).

Taking $s = -n$ in (2) and using (4) we find

$$\begin{aligned} L(-n, \chi) &= k^n \sum_{r=1}^{k-1} \chi(r) \zeta\left(-n, \frac{r}{k}\right) = k^n \sum_{r=1}^{k-1} \chi(k-r) \zeta\left(-n, \frac{k-r}{k}\right) \\ &= k^n \chi(-1) (-1)^{n+1} \sum_{r=1}^{k-1} \chi(r) \zeta\left(-n, \frac{r}{k}\right) \\ &= -\chi(-1) (-1)^n L(-n, \chi). \end{aligned}$$

If $\chi(-1) = (-1)^n$ we get $L(-n, \chi) = -L(-n, \chi)$, which proves (1).

REFERENCES

1. Tom M. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Math., Springer-Verlag, New York, 1976.
2. Karl Prachar, *Primzahlverteilung*, Die Grundlehren der Math. Wissenschaften, Band 91, Springer-Verlag, Berlin, 1957.

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125