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LATTICES ALL OF WHOSE CONGRUENCES ARE NEUTRAL[1]

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ABSTRACT. We derive a necessary condition for all congruences on a lattice to be neutral, and we show that a stronger condition of the same type characterizes relatively complemented lattices. We also find a condition necessary and sufficient for all congruences to be neutral.

1. Introduction. G. Gratzer [**4**] posed the following problem.

PROBLEM 1.1 (PROBLEM III.7 of [**4**]). Develop structure theorems for lattices all of whose congruences are standard, distributive or neutral.

This paper gives a solution to this problem for the case of neutral congruences.

First we derive a necessary condition on lattices, for all of whose congruences to be neutral, which is stronger than that given by Iqbalunnisa [**5**]. Further, using a much stronger condition, a characterization of relatively complemented lattices is obtained. Finally, a complete solution to the last part of the problem is given by another approach.

For basic notations and results we refer the reader to G. Gratzer [**4**].

2. Case of neutral congruences.

THEOREM 2.1. *A necessary condition for all congruence relations of a lattice L to be neutral is that L satisfies the following condition.*

(C) *For $a,b,c,d \in L$, $a > b$, $c > d$, $a/b \approx_w c/d$ implies the existence of $c_1 \in L$, $c \geq c_1 > d$, such that $c_1/d \approx_w a/b$.*

PROOF. Let all congruence relations of L be neutral and $a/b \approx_w c/d$, $a > b$, $c > d$, $a,b,c,d \in L$. Then there exists a neutral ideal I of L such that $\Theta(a,b) = \Theta[I]$.

Now $a \equiv b$ $(\Theta[I])$ and I is a standard ideal and hence by a theorem of [**4**] there exists $i \in I$ such that $a = b \vee i$.

But $i/b \wedge i \nearrow a/b \approx_w c/d$ implies $i/b \wedge i \approx_w c/d$. Clearly i, $b \wedge i \in I$ and I is a dually distributive ideal. Hence, by a corollary of [**2**] there exist $a_1, b_1 \in I$ and $c_1 \in L$, $c \geq c_1 > d$, such that $a_1/b_1 \nearrow c_1/d$. But as $a_1, b_1 \in I$ and $\Theta[I] = \Theta a/b$, $a_1 \equiv b_1$ $(\Theta a/b)$, which implies $c_1 \equiv d$ $(\Theta a/b)$. Hence, there exists a finite sequence of elements $d = d_1 < d_2 < \cdots < d_n = c_1$ such that $d_{i+1}/d_i \approx_w a/b$ for $i = 0, 1, \ldots, n-1$. In particular, $d_2/d \approx_w a/b$, where $d < d_2 \leq c$, which proves the theorem.

Since (C) implies weak modularity for any lattice L, the result of [**5**] follows immediately. In fact, (C) is stronger than weak modularity, for, even the three element chain does not satisfy (C).

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It is interesting to note that relatively complemented lattices always satisfy (C). In fact, a condition stronger than (C) characterizes these lattices as shown in the following.

THEOREM 2.2. *A lattice L is relatively complemented iff $a, b, c, d \in L$, $b < a$, $d < c$, $a/b \overset{n}{\approx}_w c/d$ imply $a/b \overset{n}{\approx} c'/d$, where $d < c' \leq c$.*

PROOF. Let L be relatively complemented and let

$$a/b = e_0/f_0 \sim_w e_1/f_1 \sim_w \cdots \sim_w e_n/f_n = c/d.$$

The inductive assumption for $n-1$ weak perspectivities implies the existence of $e'_{n-1} \in L$ such that

$$a/b \overset{n-1}{\approx} e'_{n-1}/f_{n-1}, \qquad f_{n-1} < e'_{n-1} \leq e_{n-1}.$$

Also, using the assumption for the dual of L, there exists $f'_{n-1} \in L$ such that $a/b \underset{n-1}{\approx} e_{n-1}/f'_{n-1}$, $f_{n-1} \leq f'_{n-1} < e_{n-1}$.

Case (1). Suppose $e_{n-1}/f_{n-1} \nearrow_w c/d$. Then

$$a/b \overset{n-1}{\approx} e'_{n-1}/f_{n-1} \nearrow c'/d, \quad \text{where } d < c' = e'_{n-1} \vee d \leq c.$$

Hence $a/b \overset{n}{\approx} c'/d$, $d < c' \leq c$.

Case (2). If $e_{n-1}/f_{n-1} \searrow_w c/d$, consider a relative complement c' of $c \wedge f'_{n-1}$ in $[d, c]$.

Now $e_{n-1}/f'_{n-1} \searrow c/c \wedge f'_{n-1} \searrow c'/d$ and therefore $e_{n-1}/f'_{n-1} \searrow c'/d$, where $d < c' \leq c$.

Hence $a/b \overset{n-1}{\approx} e_{n-1}/f'_{n-1} \searrow c'/d$. Consequently, $a/b \overset{n}{\approx} c'/d$, where $d < c' \leq c$, which proves the necessity part by induction.

Conversely, if $a > c > b$, $a, b, c \in L$, then clearly $a/c \searrow_w a/b$. Hence, there exists $c' \in L$ such that $a/c \sim c'/b$. Clearly $a/c \searrow c'/b$ as $c > b$, which implies c' is a relative complement of c in $[b, a]$ and hence the theorem follows.

However, condition (C) is not sufficient for all congruences of a lattice to be neutral as it is already known that a homomorphism kernel of a relatively complemented lattice need not be neutral (see [**6**]). The next theorem gives a solution to Problem 1.1 by another approach.

THEOREM 2.3. *Let L be any lattice. Then the following conditions are equivalent.*

(1) *All congruences of L are neutral.*

(2) *L has a zero and satisfies the condition: $x \leq y \vee z$, $x, y, z \in L$, implies the existence of $a \in L$ such that $a \vee x = (a \wedge y) \vee (a \wedge z) \vee (x \wedge y)$, $a \equiv 0$ $(\Theta(x, x \wedge y))$.*

(3) *L has a zero and satisfies the condition: $x \leq y \vee z$, $x, y, z \in L$, implies the existence of $a \in L$ such that $a \vee x = (a \wedge z) \vee (y \wedge (a \vee x))$, $a \equiv 0$ $(\Theta(x, x \wedge y))$.*

PROOF. (1) implies (2). Clearly L must have a zero, for the least congruence $\omega = \Theta[\{0\}]$.

Let $x \leq y \vee z$. Then $\Theta(x, x \wedge y) = \Theta[I]$, where I is a neutral ideal. Since I is standard, by a theorem of [**4**], there exists $a_1 \in I$ such that $x = (x \wedge y) \vee a_1$.

Now $a_1 \leq y \vee z$, $a_1 \in I$, and I is a dually distributive ideal. Hence, by a corollary of [**1**], there exists an $a \in I$ such that $a_1 \leq a = (a \wedge y) \vee (a \wedge z)$. But then

$$a \vee (x \wedge y) = (a \vee a_1) \vee (x \wedge y) = a \vee (a_1 \vee (x \wedge y)) = a \vee x.$$

Thus

$$a \vee x = a \vee (x \wedge y) = (a \wedge y) \vee (a \wedge z) \vee (x \wedge y).$$

Also $a \in I$, and therefore, $a \equiv 0$ $(\Theta(x, x \wedge y))$, which proves (2).

(2) implies (3). From (2), given $x, y, z \in L$, $x \leq y \vee z$, there exists $a \in L$ such that $a \equiv 0$ $(\Theta(x, x \wedge y))$ and $a \vee x = (a \wedge y) \vee (a \wedge z) \vee (x \wedge y)$.

Now

$$a \vee x \geq (a \wedge z) \vee (y \wedge (a \vee x)) \geq (a \wedge y) \vee (a \wedge z) \vee (x \wedge y) = a \vee x.$$

Hence (3) follows.

(3) implies (1). Let $x, y \in L$, $x \geq y$. Then $x = y \vee x$. So by (3) with $z = x$ there exists an $a \in L$ such that $a \vee x = (a \wedge x) \vee (y \wedge (a \vee x))$, $a \equiv 0$ $(\Theta(x, x \wedge y))$. But this implies $x = a \vee y$ with $a \equiv 0$ $(\Theta x/y)$. Hence, by a remark of [**3**], all congruence relations of L are standard.

Now it suffices to show that all standard ideals of L are neutral. Let I be a standard ideal of L and $x \leq y \vee z$, $x \in I$, $y, z \in L$. Then by (3), there exits $b \in L$ such that $b \vee x = (b \wedge z) \vee (y \wedge (b \vee x))$, $b \equiv 0$ $(\Theta(x, x \wedge y))$. Since $x, x \wedge y, 0 \in I$ and I is a homomorphism kernel, we get $b \in I$. Hence $b \vee x \in I$.

Further

$$((b \vee x) \wedge y) \vee ((b \vee x) \wedge z) \geq (b \wedge z) \vee (y \wedge (b \vee x)) = b \vee x.$$

Hence

$$b \vee x = ((b \vee x) \wedge y) \vee ((b \vee x) \wedge z).$$

Puting $a = b \vee x$, we get $x \leq a = (a \wedge y) \vee (a \wedge z)$ with $a \in I$. Thus, by a corollary of [**2**], I is a dually distributive ideal, which completes the proof.

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