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# ON THE CONVOLUTION EQUATIONS IN THE SPACE OF DISTRIBUTIONS OF $L^p$-GROWTH

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ABSTRACT. We consider convolution equations in the space $D'_{L^p}$, $1 \leqslant p \leqslant \infty$, of distributions of $L^p$-growth, i.e. distributions which are finite sums of derivatives of $L^p$-functions (see [4, 7]). Our main results are to find a condition for convolution operators to be hypoelliptic in $\mathscr{D}'_{L^\infty}$ in terms of their Fourier transforms and to show that the same condition is working for the solvability of convolution operators in the tempered distribution space $\mathscr{S}'$ and $\mathscr{D}'_{L^p}$.

**Preliminary.** We recall the basic facts about the spaces $\mathscr{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, and $\mathscr{S}'$, which we need in this paper. For the proof we refer to [4, 7].

*The space* $\mathscr{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$. Let $\mathscr{D}'_{L^p}$ be the space of all $C^\infty$-functions $\phi$ in $\mathbf{R}^n$ such that $\mathscr{D}^\alpha\phi$, for all $\alpha \in \mathbf{N}^n$, is in $L^p(\mathbf{R}^n)$ equipped with the topology generated by countable norms

$$\|\phi\|_{m,p} = \left\{ \sum_{|\alpha| \leqslant m} \|D^\alpha\phi\|^p_{L^p} \right\}^{1/2}, \qquad m \in \mathbf{N}, 1 \leqslant p < \infty,$$

and

$$\|\phi\|_{m,\infty} = \sup_{|\alpha| \leqslant m} \|D^\alpha\phi\|_{L^\infty}, \qquad m \in \mathbf{N}.$$

Then it is obviously a Fréchet space and a normal space of distributions in $\mathbf{R}^n$. We also have $C_c^\infty \subset \mathscr{D}_{L^p} \subset \mathscr{D}'$ with continuous injections.

We denote by $\mathscr{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, the dual of $\mathscr{D}_{L^q}$, where $1/p + 1/q = 1$ and these duals are subspaces of the space of distributions in $\mathbf{R}^n$. A distribution $T$ is in $\mathscr{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, if and only if there is an integer $m(T) > 0$ such that

$$T = \sum_{|\alpha| \leqslant m} D^\alpha f_\alpha, \qquad \alpha \in \mathbf{N}^n, \tag{1}$$

where the $f_\alpha$'s are bounded continuous functions belonging to $L^p(\mathbf{R}^n)$. Moreover, if $p < \infty$, each $f_\alpha$ converges to zero at infinity.

The Fourier transform of a function in $\mathscr{D}_{L^1}$ is a continuous function rapidly decreasing at infinity and also the Fourier transform of a distribution in $\mathscr{D}'_{L^1}$ is a continuous function slowly increasing at infinity.

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*The space $\mathscr{S}'$.* Let $\mathscr{S}$ be the space of all $C^\infty$-functions $\phi$ in $\mathbf{R}^n$ such that

$$\sup_{|\alpha| \leqslant k,\, x \in \mathbf{R}^n} (1 + |x|)^k |D^\alpha \phi(x)| < \infty, \qquad k = 0,1,2,\ldots,$$

equipped with the topology generated by these countable norms. We denote by $\mathscr{S}'$ the dual of $\mathscr{S}$. The Fourier transformation is now an isomorphism of $\mathscr{S}$ onto itself and of $\mathscr{S}'$ onto $\mathscr{S}'$.

The space $\mathscr{O}_c' (\mathscr{S}' : \mathscr{S}')$ of convolution operators in $\mathscr{S}'$ consists of distributions $S \in \mathscr{S}'$ satisfying one of the following equivalent conditions:

(i) Given any $k = 1, 2, \ldots$, $S$ can be represented in the form

$$S = \sum_{|\alpha| \leqslant m} D^\alpha f_\alpha, \tag{2}$$

where $f_\alpha$, $|\alpha| \leqslant m$, are continuous functions in $\mathbf{R}^n$ such that

$$f_\alpha(x) = O\big((1 + |x|)^{-k}\big) \quad \text{as } |x| \to \infty.$$

(ii) For every $\phi$ in $\mathscr{S}$, $S * \phi$ is in $\mathscr{S}$. Moreover, the mapping $\phi \to S * \phi$ of $\mathscr{S}$ into $\mathscr{S}$ is continuous.

The Fourier transform $\hat{S}$ of a distribution $S$ in $\mathscr{O}_c' (\mathscr{S}' : \mathscr{S}')$ is a $C^\infty$-function with the following property: For every multi-index $\alpha$ there exists a nonnegative integer $l$ such that

$$D^\alpha \hat{S}(\xi) = O\big((1 + |\xi|)^l\big) \quad \text{as } |\xi| \to \infty. \tag{3}$$

We denote by $\mathscr{O}_M (\mathscr{S}' : \mathscr{S}')$ the space of all $C^\infty$-functions with the above property (3). They are multiplication operators in $\mathscr{S}'$ and the Fourier transformation is an isomorphism of $\mathscr{O}_c' (\mathscr{S}' : \mathscr{S}')$ onto $\mathscr{O}_M (\mathscr{S}' : \mathscr{S}')$ (see [7, Volume II]).

**Hypoelliptic convolution equations in the space $\mathscr{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$.** In [**10**], Zielézny showed how to define, in a general manner, hypoelliptic and entire elliptic convolution operators in subspace of the space of distributions. He also characterized hypoelliptic and entire elliptic convolution operators in the space $\mathscr{S}'$ of tempered distributions. In [**6** and **12**], he studied hypoelliptic convolution operators in the space of distributions of exponential growth of polynomial power and, in [**5**], Pahk studied the same problem in the space of distributions of generalized exponential growth introduced in [**2**].

In this paper it can be seen that for a distribution $S$ in $\mathscr{O}_c'$ the hypoellipticity of the convolution operator $S$ in the space of tempered distributions is equivalent to the hypoellipticity in the space of bounded distributions. We define hypoelliptic convolution operators in $\mathscr{D}'_{L^\infty}$ as follows: A distribution $S$ in $\mathscr{D}'_{L^1}$ is said to be hypoelliptic in $\mathscr{D}'_{L^\infty}$, if every solution $U$ in $\mathscr{D}'_{L^\infty}$ of the convolution equation

$$S * U = V \tag{4}$$

is in $\mathscr{D}_{L^\infty}$, when $V$ is in $\mathscr{D}_{L^\infty}$; in that case equation (1) is also called hypoelliptic in $\mathscr{D}'_{L^\infty}$. Since the space of convolution operators in $\mathscr{D}'_{L^\infty}$ is $\mathscr{D}'_{L^1}$, hypoelliptic convolution operators in $\mathscr{D}'_{L^\infty}$ has to be characterized in $\mathscr{D}'_{L^1}$. Because of lack of differentiability of their Fourier transforms there are some difficulties to achieve our goal. In this paper we only consider subclasses of $\mathscr{D}'_{L^1}$, containing $\mathscr{O}'_c$, whose Fourier transforms have certain order derivatives and increase slowly at infinity. In this class we can characterize hypoelliptic convolution operators in $\mathscr{D}'_{L^\infty}$. But we have an example of hypoelliptic convolution operators in $\mathscr{D}'_{L^\infty}$ which is not in this class.

We now establish a necessary and sufficient condition for a convolution operator to be hypoelliptic in $\mathscr{D}'_{L^\infty}$. The result is proved only for a subclass of convolution operators in $\mathscr{D}'_{L^\infty}$ and the proof is based on an idea similar to that used in [**10** and **12**].

DEFINITION. $S \in \mathscr{D}'_{L^1}$ is said to be of class $H_m$ if the Fourier transform $\hat{S}$ is a $C^m$-function in $\mathbf{R}^n$ and $D^\alpha S$, $|\alpha| \leqslant m$, are slowly increasing at infinity.

The fact that the Fourier transform is a topological isomorphism from $\mathscr{O}'_c$ onto $\mathscr{O}_M$ (see [**1**, Chapter VII]) implies that every distribution in $\mathscr{O}'_c$ is of class $H_m$. This class $H_m$ of distributions in $\mathscr{D}'_{L^1}$ will be used for our study of hypoellipticity in $\mathscr{D}'_{L^\infty}$. We begin with a lemma.

LEMMA. *Let $S$ be a distribution whose Fourier transform is of the form*

$$\hat{S} = \sum_{j=1}^{\infty} a_j \delta(\xi_j), \tag{5}$$

*where the $\xi_j$ satisfy the condition*

$$|\xi_j| > 2|\xi_{j-1}| > 2^j, \qquad j = 1, 2, \ldots, \tag{6}$$

*and the $a_j$ are complex numbers such that*

$$|a_j| = O\left(|\xi_j|^\mu\right) \quad \text{as } j \to \infty \tag{7}$$

*for some $\mu$; then the series in* (5) *converges in $\mathscr{D}'_{L^\infty}$. We assert that $S \in \mathscr{D}_{L^\infty}$ if and only if*

$$|a_j| = O\left(|\xi_j|^{-\nu}\right) \quad \text{as } j \to \infty \tag{8}$$

*for every $\nu > 0$.*

PROOF. Using the fact that, for $\phi \in \mathscr{D}_{L^1}$

$$|\xi^\alpha \hat{\phi}(\xi)| \leqslant \|D^\alpha \phi\|_{L^1}, \qquad \alpha \in \mathbf{N}^n,$$

the Fourier transforms of functions in a bounded set in $\mathscr{D}_{L^1}$ are uniformly $O(|\xi|^{-\nu})$ as $|\xi| \to \infty$, for every $\nu > 0$. Therefore the series $S = \sum_{j=1}^{\infty} a_j e^{i\langle x, \xi_j\rangle}$ converges in $\mathscr{D}'_{L^\infty}$. If the $a_j$ satisfy the condition (8), then the last series and all its term-by-term derivatives converge uniformly in $\mathbf{R}^n$. Consequently, $S$ is a $C^\infty$-function bounded together with its derivatives and so belongs to $\mathscr{D}_{L^\infty}$. The converse proof is exactly the same in [**10**].

We are now in a position to prove our main theorem.

THEOREM 1. *Let $S$ be a distribution in $\mathscr{D}'_{L^1}$ which is of class $H_m$, $m > n$. Then $S$ is hypoelliptic in $\mathscr{D}'_{L^\infty}$ if and only if its Fourier transform satisfies the following condition: There are constants $a$ and $M$ such that*

$$|\hat{S}(\xi)| \geqslant |\xi|^a \quad \text{for } \xi \in \mathbf{R}^n \text{ and } |\xi| \geqslant M. \tag{9}$$

PROOF. Suppose that the condition (9) is not satisfied. Then there exists a sequence $\xi_j$ in $\mathbf{R}^n$ defined as in the Lemma and such that

$$|\hat{S}(\xi_j)| < |\xi_j|^{-j}, \qquad j = 1, 2, \ldots. \tag{10}$$

Then the series

$$U = \sum_{j=1}^{\infty} e^{i\langle x, \xi_j\rangle}$$

converges in $\mathscr{D}'_{L^\infty}$, but by the Lemma $U$ is not in $\mathscr{D}_{L^\infty}$. On the other hand,

$$S * U = \sum_{j=1}^{\infty} \hat{S}(\xi_j) e^{i\langle x, \xi_j\rangle},$$

and applying the Lemma we conclude that $S * U$ is in $\mathscr{D}_{L^\infty}$. Thus $S$ is not hypoelliptic in $\mathscr{D}'_{L^\infty}$.

Conversely, let us take a $C^\infty$-function $\psi$ in $\mathbf{R}^n$ such that

$$\psi(\xi) = \begin{cases} 1 & \text{for } |\xi| < M, \\ 0 & \text{for } |\xi| > M + 1, \end{cases}$$

where $M$ is the constant in (9). Then we define the Fourier transform $\hat{P}$ of $P$ by the formula

$$\hat{P}(\xi) = \begin{cases} 0 & \text{for } |\xi| < M, \\ \dfrac{1 - \psi(\xi)}{\hat{S}(\xi)} & \text{for } |\xi| \geqslant M. \end{cases}$$

Obviously $\hat{S}$ is a $C^m$-function slowly increasing together with its derivatives up to the $m$th order. From the fact that $S$ is of class $H_m$ and (9) we can choose a positive integer $k$ so large that

$$\hat{Q}(\xi) = \frac{1}{\left(1 + |\xi|^2\right)^k} \hat{P}(\xi)$$

and $D^\alpha \hat{Q}(\xi)$, $|\alpha| \leqslant m$, are in $L^1(\mathbf{R}^n)$ and vanish at infinity, which follows from the iterated "chain rule"

$$\partial^\alpha\left(\frac{1}{\hat{S}}\right) = \sum \frac{\pi_1^k \partial^{\alpha_j} \hat{S}}{\hat{S}^{k+1}} C_{\alpha_1 \cdots \alpha_k}; \qquad \alpha_1 + \cdots + \alpha_k = \alpha. \tag{11}$$

Then we have, applying integration by parts,

$$\begin{aligned} |Q(x)| &= \left| \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{i\langle x, \xi\rangle} \hat{Q}(\xi)\, d\xi \right| \\ &= \left| \frac{1}{(2\pi)^{n/2}} \frac{1}{\left(1 + |x|^2\right)^{m/2}} \int_{\mathbf{R}^n} e^{i\langle x, \xi\rangle} (1 - \Delta)^{m/2} \hat{Q}(\xi)\, d\xi \right| \\ &< C \frac{1}{\left(1 + |x|^2\right)^{m/2}} \quad \text{for some constant } C. \end{aligned}$$

Therefore $Q(x)$ is an $L^1$-function, and so the distribution

$$P = \left(1 + D_1^2 + \cdots + D_n^2\right)^k Q(x)$$

is in $\mathscr{D}'_{L^1}$. Furthermore $\hat{S}(\xi)\hat{P}(\xi) = 1 - \psi(\xi)$, whence, passing to the inverse Fourier transform, we see that $P$ is a rapidly decreasing parametrix for $S$ with $\hat{W} = \psi$, that is,

$$S * P = \delta - W. \tag{12}$$

Now assume that $S * U = V$, where $V \in \mathscr{D}_{L^\infty}$ and $U \in \mathscr{D}'_{L^\infty}$. Then, making use of (12), we can write

$$\begin{aligned} U = U * \delta &= U * (S * P) + U * W \\ &= (U * S) * P + U * W \\ &= V * P + U * W. \end{aligned}$$

It is obvious that $V * P$ and $U * W$ are in $\mathscr{D}_{L^\infty}$, so that $U$ is in $\mathscr{D}_{L^\infty}$.

COROLLARY. *With the same hypothesis of $S$ in the theorem*, (9) *implies that every solution $U$ in $\mathscr{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, of the equation* (4) *is in $\mathscr{D}_{L^p}$ whenever $V$ is in $\mathscr{D}_{L^p}$.*

PROOF. Viewing the proof of sufficiency of the theorem, $P$ is in $\mathscr{D}'_{L^1}$ and $U = V * P + U * W$. We can easily see the $\mathscr{D}_{L^p} * \mathscr{D}'_{L^1} \subset \mathscr{D}_{L^p}$ and so $U$ is in $\mathscr{D}_{L^p}$.

If the given convolution operator $S$ is in $\mathscr{D}'_{L^1}$, then we have the following weak version of the regularity theorem.

THEOREM 2. *If a distribution $S$ in $D'_{L^1}$ satisfies the condition* (9), *then every solution $U$ in $\mathscr{D}'_{L^\infty}$ of the equation* (1) *with $V \in \mathscr{D}_{L^2}$ is in $\mathscr{D}_{L^\infty}$.*

PROOF. Applying the same argument as in Theorem 1, we construct the continuous function $\hat{P}(\xi)$ slowly increasing at infinity, and so we find a positive integer $k$ so large that

$$\hat{Q}(\xi) = \frac{1}{\left(1 + |\xi|^2\right)^k} \hat{p}(\xi)$$

is in $L^2(\mathbf{R}^n)$. By Plancherel's theorem, $Q(x)$ is in $L^2(\mathbf{R}^n)$, and so the distribution $P = (1 + D_1^2 + \cdots + D_n^2)^k Q$ is in $\mathscr{D}'_{L^2}$. Also, we have

$$U = U * \delta = V * P + U * W.$$

Since $V$ is in $\mathscr{D}_{L^2}$, $V * P$ and $U * W$ are in $\mathscr{D}_{L^\infty}$, so that $U$ is in $\mathscr{D}_{L^\infty}$.

Combining Theorem 1 with the results of **[10]** we can state

THEOREM 3. *Let $S$ be a distribution in $\mathscr{O}'_c$. Then the following are equivalent*:

(a) *$S$ is hypoelliptic in $\mathscr{S}'$.*

(b) *$S$ is hypoelliptic in $\mathscr{D}'_{L^\infty}$.*

(c) *There exist constants $a$ and $M$ such that*

$$|\hat{S}(\xi)| \geqslant |\xi|^a \quad \textit{for } \xi \in \mathbf{R}^n \textit{ and } |\xi| \geqslant M.$$

We now give two examples of hypoelliptic convolution operators, one of which is not of class $H_m$.

EXAMPLE 1. Let $S = e^{-|t|}$ in $R^1$. Since $\hat{S}(\xi) = 1/(1 + \xi^2)$, it is in $\mathcal{O}_c'$ and satisfies the condition (9). Therefore, it is hypoelliptic in $\mathcal{S}'$ and $\mathcal{D}_{L^\infty}'$.

EXAMPLE 2. Taking $S = 1/(1 + x^2) + \delta$ in $R^1$, $\hat{S}(\xi) = e^{-|\xi|} + 1$ is not a $C^1$-function in $\mathbf{R}^1$ and satisfies the condition (9) in Theorem 1 with $a = -1$ and $M = 1$. From the fact that $1/(1 + x^2)$ is in $\mathcal{D}_{L^1}$ and $\mathcal{D}_{L^1} * \mathcal{D}_{L^\infty}' \subset \mathcal{D}_{L^\infty}$, $S$ is a hypoelliptic convolution operator in $\mathcal{D}_{L^\infty}'$.

REMARK. We can easily see that the convolution operator $S$ in $\mathcal{O}_c'$ is solvable in $\mathcal{O}_c'$ if and only if $\hat{S}$ satisfies the property (9) and has no zero in $\mathbf{R}^n$ and also characterizes the solvability in $\mathcal{D}_{L^1}'$ because $S$ is actually invertible in $\mathcal{O}_c'$. Therefore, the convolution operators in $\mathcal{O}_c'$ which have the above properties are solvable in the spaces $\mathcal{D}_{L^p}'$, $1 \leqslant p \leqslant \infty$, and $\mathcal{S}'$, but the converse does not hold in general. We still leave the problems such as what condition guarantees the hypoellipticity in $\mathcal{D}_{L^\infty}'$ for the general convolutors and the solvability in $\mathcal{D}_{L^p}'$, $1 \leqslant p \leqslant \infty$, and $\mathcal{S}'$ without invertibility.

COMMENT. We appreciate the referee for various suggestions to reform our paper. He also suggested to study the relation between the number $m$ in the condition $(H_m)$ and the number $p$ which the Fourier transform of $\hat{S}(\xi)^{-1}(1 + |\xi|^2)^{-k}$, for sufficiently large $k$, is an $L^p$-convolutor.

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