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# AN EXAMPLE IN THE THEORY OF HYPERCONTRACTIVE SEMIGROUPS

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ABSTRACT. Let $L = x(d^2/dx^2) + (1 - x)(d/dx)$ on $C_c((0, \infty))$ be the Laguerre operator. It is shown that for $t > 0$, and $1 < p < q < \infty$, $e^{tL}$: $L^p(e^{-x}\,dx) \to L^q(e^{-x}\,dx)$ has norm 1 if and only if $e^{-t} \leqslant (p - 1)/(q - 1)$ and the corresponding logarithmic Sobolev constant is not equal to $2/\lambda$, where $\lambda$ is the smallest nonzero eigenvalue of $L$.

Let $(E, \mathscr{F}, m)$ be a probability space and $\{P_t\colon t > 0\}$ a conservative Markov semigroup on $B(E)$ for which $m$ is a reversible measure (i.e. for each $t > 0$, $P_t$ is symmetric on $L^2(m)$). Then, as an easy application of Jensen's inequality, $\|P_t\|_{L^p(m)\to L^p(m)} \leqslant 1$ for all $t > 0$ and $p \in [1, \infty]$. In particular, each $P_t$ admits a unique extension $\overline{P}_t$ as a bounded operator on $L^2(m)$ and $\{\overline{P}_t\colon t > 0\}$ is a semigroup of selfadjoint contractions. A well-studied example of this situation is the *Ornstein-Uhlenbeck semigroup* $\{\Gamma_t^{(d)}\colon t > 0\}$ on $B(R^d)$: $E = R^d$, $m(dx) = \gamma^{(d)}(dx) = g^{(d)}(1, x)\,dx$, and $P_t = \Gamma_t^{(d)}$ is given by

$$\Gamma_t^{(d)} f(x) = \int g^{(d)}(1 - e^{-2t}, y - e^{-t}x) f(y)\,dy$$

where $g^{(d)}(\tau, \xi) = (2\pi\tau)^{-d/2}\exp(-|\xi|^2/2\tau)$, $(\tau, \xi) \in (0, \infty) \times R^d$. In connection with his work on constructive field theory, E. Nelson [**2**] discovered that $\{\Gamma_t^{(d)}\colon t > 0\}$ enjoys a *hypercontractivity* property. Namely, he showed that for given $1 < p < q < \infty$, $\|\Gamma_t^{(d)}\|_{L^p(\gamma^{(d)})\to L^q(\gamma^{(d)})} \leqslant 1$ if and only if $e^{-2t} \leqslant (p - 1)/(q - 1)$. In addition, he noted that if $e^{-2t} > (p - 1)/(q - 1)$, then $\|\Gamma_t^{(d)}\|_{L^p(\gamma)\to L^q(\gamma)} = \infty$. Since Nelson's initial discovery, many other examples of hypercontractive semigroups have been found (cf. F. Weissler [**7, 8**], F. Weissler and C. Mueller [**9**], and O. Rothaus [**3–5**]). In most cases the difficult part of the analysis lies in the attempt to obtain the optimal result (i.e. the smallest $T(p, q) > 0$ such that $\|P_t\|_{L^p(m)\to L^q(m)} \leqslant 1$ for all $t \geqslant T(p, q)$). The work of L. Gross [**1**] shows that this question is closely related to that of finding the smallest $\alpha > 0$ for which the *logarithmic Sobolev inequality*

$$\int |f|^2 \log|f|^2\,dm \leqslant \alpha\mathscr{E}(f, f) + \|f\|^2_{L^2(m)} \log\|f\|^2_{L^2(m)} \tag{1}$$

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holds, where $\mathscr{E}$ denotes the Dirichlet form associated with $\{\overline{P}_t: t > 0\}$ (i.e.

$$\mathscr{E}(f,f) = \sup_{t>0} \frac{1}{t}\left(f - \overline{P}_t f, f\right)_{L^2(m)} = \lim_{t\to 0} \frac{1}{t}\left(f - \overline{P}_t f, f\right)_{L^2(m)}$$

and $\mathrm{Dom}(\mathscr{E}) \equiv \{f \in L^2(m): \mathscr{E}(f,f) < \infty\}$). Indeed, under mild conditions, Gross's analysis shows that (1) for a given $\alpha > 0$ is equivalent to

$$(2) \qquad \|P_t\|_{L^p(m)\to L^q(m)} \leqslant 1, \qquad e^{-4t/\alpha} \leqslant \frac{p-1}{q-1}.$$

(cf. D. Stroock [**6**, §9], for additional information). Further, Rothaus [**3**] has shown that the *logarithmic Sobolev constant* (i.e., the smallest $\alpha$ for which (1) holds) must be at least $2/\lambda$, where

$$(3) \qquad \lambda = \inf\left\{\mathscr{E}(f,f): \|f\|_{L^2(m)} = 1 \text{ and } \int f\,dm = 0\right\}$$

is the gap between 0 and the rest of the spectrum of the generator $\{\overline{P}_t: t > 0\}$. For the most part, the technique adopted for proving optimality has been to prove that (1) holds with $\alpha = 2/\lambda$ (cf. [**9**]).

The main purpose of this note is to provide a simple example for which the hypercontractivity constant is not $2/\lambda$. To this end, take: $E = [0,\infty)$, $m(d\rho) = e^{-\rho}\,d\rho$, and for locally bounded measurable $f$: $[0,\infty) \to R^1$ having subexponential growth at $\infty$, define $P_t f$ so that

$$(4) \qquad P_t f(\rho^2/2) = \left[\Gamma_{t/2}^{(2)} \tilde{f}\right](\rho\omega), \qquad t > 0 \text{ and } \rho \in [0,\infty),$$

where $\tilde{f}(x) = f(|x|^2/2)$, $x \in R^2$, and $\omega = \binom{1}{0} \in R^2$. Then the following facts about $\{P_t: t > 0\}$ are easy to check:

$$(5) \qquad \begin{array}{ll} \text{(i)} & \{P_{t|B(E)}: t > 0\} \text{ is a conservative Markov semigroup,} \\ \text{(ii)} & \text{for each } t > 0,\ P_t \text{ is symmetric on } L^2(m). \end{array}$$

(6) LEMMA. *Let $1 < p < q < \infty$ and $t > 0$ be given. If $e^{-t} \leqslant (p-1)/(q-1)$, then $\|P_t\|_{L^p(m)\to L^q(m)} \leqslant 1$. If $e^{-t} > (p-1)/(q-1)$, then $\|P_t\|_{L^p(m)\to L^q(m)} = \infty$.*

PROOF. Note that for any $r \in [1,\infty)$ and any measurable $g$: $[0,\infty) \to R^1$, $\|g\|_{L^r(m)} = \|\tilde{g}\|_{L^r(\gamma^{(2)})}$. Also, observe that for any locally bounded $f$: $[0,\infty) \to R^1$ having subexponential growth at $\infty$, $\Gamma_{t/2}^{(2)}\tilde{f} = \widetilde{P_t f}$, $t > 0$. Thus, $\|P_t\|_{L^p(m)\to L^q(m)} \leqslant 1$ is equivalent to $\|\Gamma_{t/2}^{(2)}\tilde{f}\|_{L^q(\gamma^{(2)})} \leqslant \|\tilde{f}\|_{L^p(\gamma^{(2)})}$ for all locally bounded measurable $f$: $[0,\infty) \to R^1$ which have subexponential growth at $\infty$. In particular, by Nelson's inequality, $\|P_t\|_{L^p(m)\to L^q(m)} \leqslant 1$ if $e^{-t} \leqslant (p-1)/(q-1)$. To prove that $\|P_t\|_{L^p(m)\to L^p(m)} = \infty$ if $e^{-t} > (p-1)/(q-1)$, consider the functions $f_\lambda(\rho) = \exp(2^{1/2}\lambda\rho^{1/2} - \lambda^2/2)$ for $\lambda > 0$. In view of the preceding considerations, we need only check that

$$\lim_{\lambda\to\infty} \left\|\Gamma_{t/2}^{(2)}\tilde{f}_\lambda\right\|_{L^q(\gamma^{(2)})} \Big/ \|\tilde{f}_\lambda\|_{L^p(\gamma^{(2)})} = \infty$$

when $(p-1)/(q-1) > e^{-t}$. By straightforward computation, one can easily see that

$$\left[(\pi/2)^{1/2} r\lambda\right]^{1/r} \exp(\lambda^2(r-1)/2)$$
$$\leqslant \|\tilde{f}_\lambda\|_{L^r(\gamma^{(2)})} \leqslant \left(1 + (2\pi)^{1/2} r\lambda\right)^{1/r} \exp(\lambda^2(r-1)/2)$$

for any $\lambda > 0$ and $r \in (1, \infty)$. At the same time,

$$\left[\Gamma_{t/2}^{(2)} \tilde{f}_\lambda\right](x) \geqslant \sup_{\theta \in S^1} \left[\Gamma_{t/2}^{(2)} g_{\lambda\theta}\right](x) = \sup_{\theta \in S^1} g_{\lambda e^{-t/2}\theta}(x) = \tilde{f}_{\lambda e^{-t/2}}(x),$$

where $g_\eta(x) = \exp(\eta \cdot x - |\eta|^2/2)$ for $\eta \in R^2$ and we have used the fact that $\Gamma_s^{(2)} g_\eta = g_{e^{-s}\eta}$ for all $s > 0$ and $\eta \in R^2$. After combining these, one easily arrives at the desired conclusion. Q.E.D.

To complete our analysis, we must compute the $\lambda$ associated with $\{\bar{P}_t: t > 0\}$. To this end, let $\{Y_n: n > 0\}$ be the normalized Laguerre polynomials (i.e. the normalized orthogonal polynomials on $[0, \infty)$ with respect to $m$) and define $H = \Delta - x\nabla$ on $C^\infty(R^2)$. Then, as is well known,

$$\rho \frac{d^2 Y_n}{d\rho^2}(\rho) + (1-\rho)\frac{dY_n}{d\rho}(\rho) = -nY_n(\rho), \qquad n \geqslant 0 \text{ and } \rho \in [0, \infty).$$

From this, it is an easy matter to check that

$$H\tilde{Y}_n = -2n\tilde{Y}_n, \qquad n \geqslant 0.$$

Since $\Gamma_t^{(2)} f - f = \int_0^t \Gamma_s^{(2)} Hf\, ds$, $t > 0$, for all polynomials $f: R^2 \to R^1$, we conclude that

$$\Gamma_{t/2}^{(2)} \tilde{Y}_n = e^{-nt} \tilde{Y}_n$$

and therefore that

$$P_t Y_n = e^{-nt} Y_n$$

for all $t > 0$ and $n \geqslant 0$. As an immediate consequence, we now have that

$$\bar{P}_t f = \sum_0^\infty e^{-nt} (f, Y_n)_{L^2(m)} Y_n, \qquad t > 0 \text{ and } f \in L^2(m).$$

In particular, the Dirichlet form $\mathscr{E}$ for $\{\bar{P}_t: t > 0\}$ is given by

$$\mathscr{E}(f, f) = \sum_1^\infty n (f, Y_n)^2_{L^2(m)}, \qquad f \in L^2(m),$$

and so the corresponding gap $\lambda$ is 1.

By combining Gross's analysis, Lemma (6) and the preceding, we now have the following result.

(7) THEOREM. *Let $m(d\rho) = e^{-\rho}\, d\rho$ on $[0, \infty)$ and define $P_t$, $t > 0$, by (4). Then $\{P_t: t > 0\}$ is a conservative Markov semigroup which is symmetric in $L^2(m)$. Let $\{\bar{P}_t: t > 0\}$ be the semigroup of $L^2(m)$-selfadjoint contractions determined by $\{P_t: t > 0\}$ and denote by $\mathscr{E}$ the associated Dirichlet form. Then*

$$1 = \inf\left\{\mathscr{E}(f, f): f \in L^2(m),\ \|f\|_{L^2(m)} = 1 \text{ and } \int f\, dm = 0\right\},$$

*On the other hand, the logarithmic Sobolev constant for $\mathscr{E}$ (i.e. the smallest $\alpha$ for which (2) holds) is 4.*

REMARK. The semigroup $\{P_t\colon t > 0\}$ in Theorem (7) can be described directly in terms of the *Laguerre operator*

$$L = \rho\frac{d^2}{d\rho^2} + (1-\rho)\frac{d}{d\rho} \quad \text{on } C_c^\infty((0,\infty)).$$

Indeed, $\{P_t\colon t > 0\}$ is the unique conservative Markov semigroup on $B((0,\infty))$ such that

$$P_t f - f = \int_0^t P_s L f\, ds, \qquad t \geqslant 0,$$

for all $f \in C_c^\infty((0,\infty))$. Thus there are several reasons for calling $\{P_t\colon t > 0\}$ the *Laguerre semigroup*. In this connection it is natural to suspect that the reason why, in this example, the logarithmic Sobolev constant $\alpha_0$ and the spectral gap $\lambda$ do not satisfy $\alpha_0 = 2/\lambda$ may have something to do with the way in which $L$ degenerates at 0.

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