PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 1, May 1985

SUMMING GENERALIZED CLOSED *U*-SETS FOR WALSH SERIES

KAORU YONEDA

ABSTRACT. A countable union of closed U-sets for Walsh series in certain generalized sense is again a U-set in the same sense.

1. Introduction. Let $\mu \sim \sum_{k=0}^{\infty} \hat{\mu}(k) w_k(x)$ be a Walsh series. A subset E of the dyadic group is said to be a *U-set if*

$$(1) \qquad \sum_{k=0}^{\infty} \hat{\mu}(k) w_k(x) = 0 \quad \textit{everywhere except on } E$$

implies that μ *is the zero series.*

Wade [2] proved that *if* $E_1, E_2, \ldots$ *are closed U-sets, then* $\bigcup_{k=1}^{\infty} E_k$ *is also a U-set.*

In this paper we shall generalize Wade's theorem. Let $\mathscr{A}$ be a certain class of Walsh series. A subset E of the dyadic group is said to be a *U-set for* $\mathscr{A}$ *if* $\mu \in \mathscr{A}$ *and*

$$(2) \qquad \lim_{n \to \infty} \sum_{k=0}^{2^n - 1} \hat{\mu}(k) w_k(x) = 0 \quad \textit{everywhere except on } E$$

imply that μ *is the zero series*. We have already proved in [3] that *when E is a closed subset of the dyadic group*, (1) *holds if and only if* (2) *holds and*

$$(3) \qquad \hat{\mu}(k) = o(1) \quad \textit{as } k \to \infty.$$

Therefore *a closed subset of the dyadic group is a U-set in the classical sense if and only if it is a U-set for the class of Walsh series* μ *which satisfies* (3).

When $\mathscr{A}$ satisfies the following conditions, we say that $\mathscr{A}$ *satisfies the condition* (L):

(i) *a U-set for* $\mathscr{A}$ *is of measure zero*;

(ii) *if* $\mu \in \mathscr{A}$, *then*

$$\liminf_{n \to \infty} \left| \frac{1}{2^n} \sum_{k=0}^{2^n - 1} \hat{\mu}(k) w_k(x) \right| = 0 \quad \textit{everywhere};$$

(iii) *if* μ *and* $\mu' \in \mathscr{A}$, *then* $\alpha\mu + \alpha'\mu' \in \mathscr{A}$ *for arbitrary real numbers* α *and* α', *where*

$$(\alpha\mu + \alpha'\mu') \sim \sum_{k=0}^{\infty} (\alpha\mu + \alpha'\mu')\hat{\ }(k) w_k(x)$$
$$\equiv \sum_{k=0}^{\infty} \left(\alpha\hat{\mu}(k) + \alpha'\hat{\mu}'(k)\right) w_k(x);$$

Received by the editors June 18, 1984.
1980 *Mathematics Subject Classification*. Primary 42C25.
Key words and phrases. Uniqueness, Walsh series.

(iv) *if* $\mu \in \mathscr{A}$, *then*

$$\sum_{k=0}^{\infty} \hat{\mu}(k \dot{+} j) w_k(x) \in \mathscr{A} \quad \textit{for } j = 1, 2, \ldots.$$

We shall prove the following theorem.

THEOREM 1. *When a class of Walsh series $\mathscr{A}$ satisfies the condition* (L) *and if $E_1, E_2, \ldots$ are closed U-sets for $\mathscr{A}$, then $\bigcup_{k=1}^{\infty} E_k$ is also a U-set for $\mathscr{A}$.*

2. Notations and lemmas. In this paper we shall use the following notations. Let I_n^p be the set of all 0-1 sequences, $(t_1, t_2, \ldots)$, such that $\sum_{k=1}^{n} t_k/2^k = p/2^n$. I_n^p is called *a dyadic interval of rank n*. For convenience, $I_n(x)$ denotes the dyadic interval of rank n containing x. A dyadic interval is closed and open. We refer the details of the dyadic group, Walsh functions, the operation $\dot{+}$ and so on to Fine's paper [**1**].

LEMMA 2. *When $\mathscr{A}$ satisfies the condition* (L), *if $\mu \in \mathscr{A}$ and I is a dyadic interval, then there exists a Walsh series $\mu^* \in \mathscr{A}$ which satisfies the following conditions*:

(i) $$\lim_{n \to \infty} \left| \sum_{k=0}^{2^n - 1} \hat{\mu}(k) w_k(x) - \sum_{k=0}^{2^n - 1} \hat{\mu}^*(k) w_k(x) \right| = 0 \quad \textit{on } I,$$

(ii) $$\lim_{n \to \infty} \sum_{k=0}^{2^n - 1} \hat{\mu}^*(k) w_k(x) = 0 \quad \textit{uniformly everywhere except on } I.$$

PROOF. From the hypothesis, there exist an element of the dyadic group, x_0, and an integer N such that $I = I_N(x_0)$. Set

$$\hat{\mu}^*(k) = \frac{1}{2^N} \sum_{j=0}^{2^N - 1} w_j(x_0) \hat{\mu}(k \dot{+} j)$$

for $k = 0, 1, \ldots$. Since

$$\begin{aligned}
\sum_{k=0}^{2^n - 1} \hat{\mu}^*(k) w_k(x) &= \sum_{s=0}^{2^{n-N} - 1} \sum_{k=s2^N}^{(s+1)2^N - 1} \hat{\mu}^*(k) w_k(x) \\
&= \sum_{s=0}^{2^{n-N} - 1} \left(\sum_{k=0}^{2^N - 1} \hat{\mu}^*(s2^N \dot{+} k) w_{s2^N \dot{+} k}(x) \right) \\
&= \sum_{s=0}^{2^{n-N} - 1} \sum_{k=0}^{2^N - 1} \left\{ \frac{1}{2^N} \sum_{j=0}^{2^N - 1} w_j(x_0) \hat{\mu}(s2^N \dot{+} k \dot{+} j) \right\} w_{s2^N \dot{+} k}(x) \\
&= \sum_{s=0}^{2^{n-N} - 1} \sum_{j=0}^{2^N - 1} \frac{1}{2^N} w_j(x_0) \sum_{k=0}^{2^N - 1} \hat{\mu}(s2^N \dot{+} k \dot{+} j) w_{s2^N \dot{+} k}(x) \\
&= \sum_{s=0}^{2^{n-N} - 1} \sum_{j=0}^{2^N - 1} \frac{1}{2^N} w_j(x_0) w_j(x) \sum_{k=0}^{2^N - 1} \hat{\mu}(s2^N \dot{+} k \dot{+} j) w_{s2^N \dot{+} k}(x) \times w_j(x) \\
&= \sum_{s=0}^{2^{n-N} - 1} \frac{1}{2^N} \left\{ \sum_{j=0}^{2^N - 1} w_j(x_0 \dot{+} x) \right\} \left\{ \sum_{k=s2^N}^{(s+1)2^N - 1} \hat{\mu}(k) w_k(x) \right\} \\
&= \left\{ \frac{1}{2^N} \sum_{j=0}^{2^N - 1} w_j(x_0 + x) \right\} \left\{ \sum_{k=0}^{2^n - 1} \hat{\mu}(k) w_k(x) \right\},
\end{aligned}$$

and

$$\sum_{j=0}^{2^N-1} w_j(x_0 \dotplus x) = \begin{cases} 2^N, & \text{for } x \in I_N(x_0), \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\sum_{k=0}^{2^n-1} \hat{\mu}^*(k) w_k(x) = \begin{cases} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x), & \text{for } x \in I_N(x_0), \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that μ^* satisfies the conclusion.

LEMMA 3. *If a Walsh series μ satisfies the following conditions*:

(i) $$\liminf_{n\to\infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| = 0 \quad a.e.;$$

(ii) $$\sup_n \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| < \infty \quad \textit{everywhere except on a countable set};$$

(iii) $$\liminf_{n\to\infty} \left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| = 0 \quad \textit{everywhere};$$

then μ is the zero series.

Lemma 3 is Theorem 3 in [**3**].

COROLLARY 4. *When $\mathscr{A}$ satisfies the condition* (L), *if E is a closed U-set for $\mathscr{A}$ and I is a dyadic interval which contains E, if a Walsh series $\mu \in \mathscr{A}$ satisfies the following conditions*:

(i) $$\sup_n \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| < \infty \quad \textit{everywhere on } I \setminus E;$$

(ii) $$\liminf_{n\to\infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| = 0 \quad a.e. \textit{ on } I;$$

then

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0 \quad \textit{everywhere on } E.$$

PROOF. Since E is a closed set, for $x_0 \in I \setminus E$, there exists an integer N such that $I_N(x_0) \subset I$ and

(4) $$I_N(x_0) \cap E = \varnothing.$$

Let μ^* be a Walsh series which is introduced in Lemma 2. Hence μ^* satisfies that

(5) $$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}^*(k) w_k(x) = 0 \quad \text{everywhere } \textit{on } I_N^c(x_0).$$

We shall prove that μ^* satisfies the conditions of Lemma 3. Since $\sum_{k=0}^{2^n-1}\hat{\mu}(k)w_k(x)$ and $\sum_{k=0}^{2^n-1}\hat{\mu}^*(k)w_k(x)$ are equiconvergent on $I_N(x_0)$, μ^* satisfies (5), (i) and (ii) of Lemma 3. From (i) we have

$$\sup_n \left|\sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x)\right| < \infty \quad \text{everywhere on } I_N(x_0). \tag{6}$$

On the other hand from (5), (6) holds on $I_N^c(x_0)$. Hence (6) holds everywhere. From the definition of $\hat{\mu}^*(k)$ and the hypothesis, we have $\mu^* \in \mathscr{A}$. By Lemma 3, μ^* is the zero series. Then, we have

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0 \quad \text{everywhere on } I \setminus E.$$

Let μ^{**} be a Walsh series associated with I which is introduced in Lemma 2. Since $\mu^{**} \in \mathscr{A}$ and the 2^nth partial sums of μ^{**} and μ are equiconvergent on I, we have

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}^{**}(k)w_k(x) = 0 \quad \text{everywhere } I \text{ except on } E. \tag{7}$$

On the other hand, (7) holds on I^c. Hence (7) holds everywhere except on E. Since E is a U-set for $\mathscr{A}$, μ^{**} is the zero series. Therefore, (7) holds everywhere on I. Since the 2^nth partial sums of μ and μ^{**} are equiconvergent, the proof of Corollary 4 is complete.

Lemma 5. *Let f_n, $n = 0, 1, \ldots,$ be a function which is continuous on the dyadic group, then the following set*

$$N = \left\{x\colon \limsup_{n\to\infty} |f_n(x)| = \infty\right\}$$

is empty, countable or of the second category on itself.

The proof is due to [2].

3. Proof of Theorem 1. Set $E = \bigcup_{i=1}^{\infty} E_i$. Let μ satisfy

$$\liminf_{n\to\infty} \left|\sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x)\right| = 0 \quad \text{everywhere except on } E. \tag{8}$$

Each E_i is a U-set for $\mathscr{A}$, then from (i) of (L), E_i is of measure zero. Hence E is of measure zero. Consequently μ satisfies (8) a.e. Set

$$N = \left\{x\colon \limsup_{n\to\infty} \left|\sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x)\right| = \infty\right\}.$$

Since $\sum_{k=0}^{2^n-1}\hat{\mu}(k)w_k(x)$ is a continuous function on the dyadic group, by Lemma 5, three cases arise.

Now we shall assume that N is of the second category on itself. Set $N_i = N \cap E_i$. Then, there exist a dyadic interval I and an integer i_0 such that $N \cap I \neq \varnothing$ and $N_{i_0} \cap I$ is dense in $N \cap I$. Since E_{i_0} is closed, we have $N_{i_0} = N \cap E_{i_0}$. We shall prove that

$$N \cap I = E_{i_0} \cap N \cap I \subseteq E_{i_0} \cap I. \tag{9}$$

It is obvious that $N \cap I \supseteq E_{i_0} \cap N \cap I$. If $x \in N \cap I$, then there exists a sequence of elements $\{x_k\}_k$ such that $x_k \in N_{i_0} \cap I$ and $\lim_{k\to\infty} x_k = x$. Since $x_k \in N_{i_0}$ and $x_k \in I$, we have $x_k \in E_{i_0}$. E_{i_0} is closed, therefore we have $x \in E_{i_0}$. Hence we proved the conclusion. It is obvious that $E_{i_0} \cap I$ is a closed U-set for $\mathscr{A}$ and that $E_{i_0} \cap I \subseteq I$. Assume that $x \notin E_{i_0} \cap I$. Then $x \notin N \cap I$. Hence we have

$$\sup_n \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| < \infty \quad \text{in } I \setminus (E_{i_0} \cap I) \equiv I \setminus E_{i_0}.$$

By Corollary 4, we have

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{everywhere on } I.$$

Hence we proved that

$$N \cap I = \varnothing. \tag{10}$$

(10) contradicts the assumption $N \cap I \neq \varnothing$. Therefore we have that N is not of the second category on itself. The proof is complete.

A subset E of the dyadic group is said to be a *U_1-set for $\mathscr{A}$* [3] if $\mu \in \mathscr{A}$ *and*

$$\liminf_{n\to\infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| = 0 \quad \textit{everywhere except on } E \tag{2'}$$

imply that μ is the zero series.

We can prove analogously to Theorem 1 the following theorem.

THEOREM 1′. *When a class of Walsh series $\mathscr{A}$ satisfies the condition* (L), *if $E_1, E_2, \ldots$ are closed U_1-sets for $\mathscr{A}$, then $\bigcup_{k=1}^{\infty} E_k$ is also a U_1-set for $\mathscr{A}$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSAKA PREFECTURE, SAKAI, OSAKA, JAPAN