PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 1, May 1985

ON DENSE SUBSETS OF THE MEASURE ALGEBRA

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ABSTRACT. We show that the minimal cardinality of a dense subset of the measure algebra is the same as the minimal cardinality of a base of the ideal of Lebesgue measure zero subsets of the real line.

0. Introduction. Let $(P, \leqslant)$ be a given partial ordering. A subset $D \subseteq P$ is called dense if for any $p \in P$ there exists $d \in D$ such that $d \leqslant p$. A subset D is called upward dense if D is dense in $(P, \geqslant)$. Let $\Delta(P, \leqslant)$ denote the minimal cardinality of a dense subset of $(P, \leqslant)$.

Recall that with any boolean algebra $\mathscr{C}$ we can associate a natural partial ordering $\leqslant_{\mathscr{C}}$. Let 0 and 1 denote the minimal and the maximal elements in this ordering. A dense or upward dense subset of $\mathscr{C}$ is a subset of $\mathscr{C} \setminus \{0,1\}$ which is dense or upward dense with respect to $\leqslant_{\mathscr{C}}$. Note that

$$\Delta(\mathscr{C} \setminus \{0,1\}, \leqslant_{\mathscr{C}}) = \Delta(\mathscr{C} \setminus \{0,1\}, {}_{\mathscr{C}}\geqslant).$$

We call this cardinal number $\Delta(\mathscr{C})$.

Let $\mathscr{I}$ be an ideal of subsets of a set X. A base of $\mathscr{I}$ is an upward dense subset of $(\mathscr{I}, \subseteq)$. Let $\Delta(\mathscr{I}) = \Delta(\mathscr{I}, \supseteq)$. $\mathscr{I}$ is a σ-ideal if for any countable $\mathscr{A} \subseteq \mathscr{I}$ we have $\bigcup \mathscr{I} \in \mathscr{I}$. Let $\mathscr{I}^c$ denote the dual filter $\{X \setminus A \colon A \in \mathscr{I}\}$.

In this paper $\mathscr{B}$ denotes the σ-field of Borel subsets of the Cantor set ${}^{\omega}2$ (ω denotes the set of natural numbers and 2 denotes the set $\{0,1\}$). Let $\mathscr{K}$ denote the ideal of subsets of ${}^{\omega}2$ of first Baire category. The canonical product measure on ${}^{\omega}2$ is called the Lebesgue measure on ${}^{\omega}2$. Let $\mathscr{L}$ denote the ideal of subsets of ${}^{\omega}2$ of Lebesgue measure zero. Note that both ideals $\mathscr{K}$ and $\mathscr{L}$ have Borel bases (i.e. bases contained in $\mathscr{B}$). If $\mathscr{I}$ is an ideal on ${}^{\omega}2$ then $\mathscr{I} \cap \mathscr{B}$ is an ideal in the σ-field $\mathscr{B}$. We denote the quotient boolean algebra $\mathscr{B}/(\mathscr{I} \cap \mathscr{B})$ by $\mathscr{B}/\mathscr{I}$.

A boolean algebra is σ-saturated if there is no uncountable family of pairwise disjoint elements of this algebra. Recall that the algebras $\mathscr{B}/\mathscr{K}$ and $\mathscr{B}/\mathscr{L}$ are σ-saturated. Suppose that $\mathscr{C}$ is a boolean algebra and $a \in \mathscr{C}$. Then by $\mathscr{C}_a$ we denote the boolean algebra with the universe $\{b \in \mathscr{C} \colon b \leqslant_{\mathscr{C}} a\}$ endowed with the operations canonically defined from the operations in $\mathscr{C}$.

1. The main result. We show a method of constructing a dense subset of the boolean algebra $\mathscr{B}/\mathscr{I}$ from a base of $\mathscr{I}$ for some class of ideals on ${}^{\omega}2$.

Received by the editors July 29, 1983.
1980 *Mathematics Subject Classification*. Primary 04A15; Secondary 28A05.

THEOREM 1.1. *Suppose $\mathscr{I}$ is a σ-ideal on ${}^{\omega}2$ with a Borel base and $\mathscr{B}/\mathscr{I}$ is σ-saturated. Then $\Delta(\mathscr{B}/\mathscr{I}) \leqslant \Delta(\mathscr{I}) \cdot \aleph_0$.*

PROOF. For $n \in \omega$ let $I_n = \{f \in {}^{\omega}2: f(n) = 0\}$. The family $\{I_n: n \in \omega\}$ σ-generates $\mathscr{B}$ and has the following independence property: for any σ-complete Boolean algebra $\mathscr{C}$ and a function $f: \{I_n: n \in \omega\} \to \mathscr{C}$, there exists a σ-complete homomorphism $h: \mathscr{B} \to \mathscr{C}$ which extends f (see [**3**, Theorem 31.6]).

For any $A \in \mathscr{B}$, we fix an ordinal $\alpha(A) < \omega$, and a sequence $\{B_\xi^A: \xi \leqslant \alpha(A)\}$ such that $B_{\alpha(A)}^A = A$, and for every $\xi \leqslant \alpha(A)$ we have $B_\xi^A = {}^{\omega}2 \setminus B_\eta^A$ for some $\eta < \xi$, or $B_\xi^A = \bigcup\{B_\eta^A: \eta \in X\}$ for some $X \subseteq \xi$, or $B_\xi^A = I_n$ for some $n \in \omega$.

LEMMA 1.2. *Suppose $\mathscr{J}$ is a σ-ideal on ${}^{\omega}2$ with a Borel base and $\mathscr{B}/\mathscr{J}$ is σ-saturated. Then there exists $a \in \mathscr{B}/\mathscr{J} \setminus \{0\}$ such that $\Delta((\mathscr{B}/\mathscr{J})_a) \leqslant \Delta(\mathscr{J}) \cdot \aleph_0$.*

PROOF. Let $\mathscr{A} \subseteq \mathscr{B} \cap \mathscr{J}$ be a base of $\mathscr{J}$ of minimal cardinality. Let $\mathscr{C}_0$ be the subalgebra of $\mathscr{B}$ generated by $\{B_\xi^A: A \in \mathscr{A} \wedge \xi \leqslant \alpha(A)\} \cup \{\mathscr{I}_n: n \in \omega\}$. Then $|\mathscr{C}_0| \leqslant \Delta(\mathscr{J}) \cdot \aleph_0$.

Let $\mathscr{C}$ be the complete boolean algebra for which $\mathscr{C}_0/\mathscr{J}$ is a dense subalgebra and let $\pi: \mathscr{C}_0 \to \mathscr{C}$ be the canonical embedding, i.e. $\pi(C) = [C]_{\mathscr{J}}$. Let $\tilde{\pi}: \mathscr{B} \to \mathscr{C}$ be the σ-homomorphism extending $\pi \restriction \{I_n: n \in \omega\}$. For $A \in \mathscr{A}$ and $\xi \leqslant \alpha(A)$, one can prove by induction on ξ that $\tilde{\pi}(B_\xi^A) = \pi(B_\xi^A)$. It follows that, for every $C \in \mathscr{C}_0$, we have $\tilde{\pi}(C) = \pi(C)$, i.e., $\tilde{\pi}$ extends π.

Let $\varphi: \mathscr{B}/\mathscr{J} \to \mathscr{C}$ be the σ-homomorphism defined by $\varphi([B]_{\mathscr{J}}) = \tilde{\pi}(B)$ for $B \in \mathscr{B}$. The definition is correct because $\mathscr{J} \subseteq \ker \tilde{\pi}$. Let $\mathscr{E} \subseteq \mathscr{B} \setminus \mathscr{J}$ be a maximal $\mathscr{J}$-disjoint family such that $\{[E]_{\mathscr{J}}: E \in \mathscr{E}\} \subseteq \ker \varphi$ ($\mathscr{J}$-disjoint means that $E \cap F \in \mathscr{J}$ for any different elements of $\mathscr{E}$). Then $\mathscr{E}$ is countable, so $[\bigcup\mathscr{E}]_{\mathscr{J}} \in \ker \varphi$. Let $a = [{}^{\omega}2 \setminus \bigcup\mathscr{E}]_{\mathscr{J}}$. Then $a \neq 0$ and the algebras $(\mathscr{B}/\mathscr{J})/\ker \varphi$ and $(\mathscr{B}/\mathscr{J})_a$ are isomorphic. Since $\varphi(\mathscr{B}/\mathscr{J}) = \mathscr{C}$, the algebras $(\mathscr{B}/\mathscr{J})_a$ and $\mathscr{C}$ are isomorphic. $\mathscr{C}$ has a dense subset, the cardinality of which is $\Delta(\mathscr{J}) \cdot \aleph_0$, so $(\mathscr{B}/\mathscr{J})_a$ does also. Hence, the lemma is proved.

Now suppose $\mathscr{I}$ satisfies the assumptions of the theorem. Then let

$$\mathscr{A} = \{a \in \mathscr{B}/\mathscr{I} \setminus \{0\}: \Delta((\mathscr{B}/\mathscr{I})_a) \leqslant \Delta(\mathscr{I}) \cdot \aleph_0\}.$$

We show that $\mathscr{A}$ is dense in $\mathscr{B}/\mathscr{I}$. Suppose $[B]_{\mathscr{I}} \in \mathscr{B}/\mathscr{I} \setminus \{0\}$ and let $\mathscr{J}$ be the ideal generated by $\mathscr{I}$ and ${}^{\omega}2 \setminus B$. Then $\mathscr{B}/\mathscr{J}$ is σ-saturated and $\Delta(\mathscr{J}) \leqslant \Delta(\mathscr{I})$. By the lemma there exists $a' \in \mathscr{B}/\mathscr{J} \setminus \{0\}$ such that $\Delta((\mathscr{B}/\mathscr{J})_a) \leqslant \Delta(\mathscr{J}) \cdot \aleph_0 \leqslant \Delta(\mathscr{I}) \cdot \aleph_0$. Then there exists a Borel set $A \subseteq B$ such that $a' = [A]_{\mathscr{J}}$. Let $a = [A]_{\mathscr{I}}$. Then the algebras $(\mathscr{B}/\mathscr{J})_{a'}$ and $(\mathscr{B}/\mathscr{I})_a$ are isomorphic and $a \leqslant [B]_{\mathscr{I}}$. Hence, $\mathscr{A}$ is dense in $\mathscr{B}/\mathscr{I}$.

Now let $\mathscr{A}^*$ be a maximal subfamily of $\mathscr{A}$ consisting of pairwise disjoint elements. Then $|\mathscr{A}^*| \leqslant \aleph_0$. For each $a \in \mathscr{A}^*$ let $\mathscr{D}_a$ be a dense subset of $(\mathscr{B}/\mathscr{I})_a$, the cardinality of which is at most $\Delta(\mathscr{I}) \cdot \aleph_0$. Then $\bigcup\{\mathscr{D}_a: a \in \mathscr{A}^*\}$ is a dense subset of $\mathscr{B}/\mathscr{I}$ with cardinality at most $\Delta(\mathscr{I}) \cdot \aleph_0$. Hence the theorem is proved.

2. Steinhaus property of ideals. It follows from Theorem 1.1 that $\Delta(\mathscr{B}/\mathscr{L}) \leqslant \Delta(\mathscr{L})$. We show that the reverse inequality is also true.

Let G be a group, $a \in G$ and $A, B \subseteq G$. We define $A^{-1} = \{x^{-1}: x \in A\}$, $A \cdot B = \{x \cdot y: x \in A \wedge y \in B\}$ and $A \cdot a = \{x \cdot a: x \in A\}$.

DEFINITION 2.1. Let G be a group and $\mathscr{C}$ a σ-complete field of subsets of G. A σ-complete ideal $\mathscr{I}$ on G satisfies the $\mathscr{C}$-Steinhaus property if there exist a base of $\mathscr{I}$ contained in $\mathscr{C}$ and a countable set $W \subseteq G$ such that

(i) $(\forall A \in \mathscr{I})(\forall q \in W)(A \cdot q \in \mathscr{I})$,

(ii) $(\forall A, B \in \mathscr{C} \setminus \mathscr{I})((B^{-1} \cdot A) \cap W \neq \varnothing)$.

Two well-known ideals on ${}^{\omega}2$ which satisfy the $\mathscr{B}$-Steinhaus proeprty are $\mathscr{L}$ and $\mathscr{K}$ (see [**2**, Theorem 4.8]). We give some other examples. Suppose $\mathscr{I}$ and $\mathscr{J}$ are ideals on ${}^{\omega}2$. Then

$$\mathscr{I} \times \mathscr{J} = \{X \subseteq {}^{\omega}2 \times {}^{\omega}2: \{x: \{y: \langle x, y\rangle \in X\} \notin \mathscr{J}\} \in \mathscr{I}\}.$$

Let $\mathscr{B}_2$ denote the σ-field of Borel subsets on ${}^{\omega}2 \times {}^{\omega}2$. Then we define

$$\mathscr{I} \otimes \mathscr{J} = \{X \subseteq {}^{\omega}2 \times {}^{\omega}2: (\exists Y \in \mathscr{B}_2 \cap (\mathscr{I} \times \mathscr{J}))\,(X \subseteq Y)\}.$$

It is easy to show that the ideals $\mathscr{L} \otimes \mathscr{K}$ and $\mathscr{K} \otimes \mathscr{L}$ satisfy the $\mathscr{B}_2$-Steinhaus property.

The next lemma clarifies condition (ii) of Definition 2.1.

LEMMA 2.2. *Let G be a group and $\mathscr{C}$ a σ-field of subsets of G. Let $\mathscr{I}$ be an ideal with a base contained in $\mathscr{C}$. Suppose W is a countable subset of G such that* $(\forall A \in \mathscr{I})(\forall q \in W)(A \cdot q \in \mathscr{I})$. *Then the following sentences are equivalent*:

(i) $(\forall A, B \in \mathscr{C} \setminus \mathscr{I})((B^{-1} \cdot A) \cap W \neq \varnothing)$,

(ii) $(\forall A \in \mathscr{C} \setminus \mathscr{I})(A \cdot W^{-1} \in \mathscr{I}^c)$,

(iii) $(\forall A \in \mathscr{C} \setminus \mathscr{I}^c)(\bigcap\{A \cdot q^{-1}: q \in W\} \in \mathscr{I})$.

PROOF. (ii) $\Leftrightarrow$ (iii) is obvious. (i) $\Rightarrow$ (ii). Suppose $A \in \mathscr{C} \setminus \mathscr{I}$ and $A \cdot W^{-1} \notin \mathscr{I}^c$. Then there exists $B \in \mathscr{C} \setminus \mathscr{I}$ such that $(A \cdot W^{-1}) \cap B = \varnothing$. But then $(B^{-1} \cdot A) \cap W = \varnothing$.

(ii) $\Rightarrow$ (i). Suppose $A, B \in \mathscr{C} \setminus \mathscr{I}$. Then by (ii) we have $(A \cdot W^{-1}) \in \mathscr{I}^c$; hence, $(A \cdot W^{-1}) \cap B \neq \varnothing$, so $(B^{-1} \cdot A) \cap W \neq \varnothing$.

The following lemma is an application of the Steinhaus property.

LEMMA 2.3. *Suppose $\mathscr{C}$ is a σ-field on a group G. If the ideal $\mathscr{I}$ on G satisfies the $\mathscr{C}$-Steinhaus property, then* $\Delta(\mathscr{I}) \leqslant \Delta(\mathscr{C} \setminus \mathscr{I})$.

PROOF. Let $\mathscr{A}$ be a dense subset of $(\mathscr{C} \setminus \mathscr{I}^c, \supseteq)$, the cardinality of which is $\Delta(\mathscr{C} \setminus \mathscr{I}^c, \supseteq)$. Let W be a countable subset of G given by the Steinhaus property. Let $\mathscr{A}^* = \{\bigcap\{A \cdot q^{-1}: q \in W\}: A \in \mathscr{A}\}$. Then $|\mathscr{A}^*| \leqslant |\mathscr{A}|$ and $\mathscr{A}^* \subseteq \mathscr{I}$. We show that $\mathscr{A}^*$ is a base of $\mathscr{I}$. Let $B \in \mathscr{I}$. Then $B \cdot W \in \mathscr{I}$. Hence, there exists $A \in \mathscr{A}$ such that $B \cdot W \subseteq A$. Then $B \subseteq \bigcap\{A \cdot q^{-1}: q \in W\}$. Hence, the lemma is proved.

The next lemma gives us a lower estimation of $\Delta(\mathscr{B}/\mathscr{L})$.

LEMMA 2.4. $\Delta(\mathscr{B} \setminus \mathscr{L}, \subseteq) \leqslant \Delta(\mathscr{B}/\mathscr{L})$.

PROOF. Let $\mathscr{A}$ be a dense subset of $\mathscr{B}/\mathscr{L} \setminus \{0\}$ with cardinality $\Delta(\mathscr{B}/\mathscr{L})$. We may assume $\mathscr{A} = \{[A]: A \in \mathscr{A}^*\}$ and, for each $A \in \mathscr{A}^*$ and every open set U, if $A \cap U \neq \varnothing$, then $A \cap U \notin \mathscr{L}$. We show that $\mathscr{A}^*$ is dense in $(\mathscr{B} \setminus \mathscr{L}, \subseteq)$. Suppose $B \in \mathscr{B} \setminus \mathscr{L}$. Let $C \subseteq B$ be a closed set such that $C \notin \mathscr{L}$. Then there exists $A \in \mathscr{A}^*$ such that $[A] \leqslant [C]$. But ${}^{\omega}2 \setminus C$ is an open set, hence $A \subseteq C$.

Note that $\Delta(\mathscr{B} \setminus \mathscr{I}, \subseteq) = \Delta(\mathscr{B} \setminus \mathscr{I}^c, \supseteq)$. Hence from Theorem 1.1 and Lemmas 2.3 and 2.4 we obtain

THEOREM 2.5. $\Delta(\mathscr{B}/\mathscr{L}) = \Delta(\mathscr{L})$.

3. Generalizations and comments. (1) We show that in the formulation of Theorem 1.1 the assumption of the existence of a Borel base is essential.

THEOREM 3.1. *If the theory ZFC is consistent, then the theory ZFC* + $2^{\aleph_0} = \aleph_2$ + "*there exists a σ-ideal* $\mathscr{I}$ *on* ${}^\omega 2$ *such that* $\mathscr{B}/\mathscr{I}$ *is σ-saturated*, $\Delta(\mathscr{B}/\mathscr{I}) = \aleph_2$ *and* $\Delta(\mathscr{I}) = \aleph_1$" *is consistent.*

PROOF. Let us add to a model of ZFC + $2^{\aleph_0} = \aleph_1$, by means of a generic extension, a sequence $\{r_\alpha : \alpha < \omega_2\}$ of independent random reals. Let $X = \{r_\alpha : \alpha < \omega_1\}$. Then for every set $A \in \mathscr{L}$, we have $|A \cap X| \leqslant \aleph_0$.

Let $\mathscr{J} = \{A \subseteq {}^\omega 2 : |A \cap X| \leqslant \aleph_0\}$. Then $\Delta(\mathscr{J}) = \aleph_1$. Let $\mathscr{A}$ be a maximal family of pairwise disjoint sets from $\mathscr{B} \setminus \mathscr{L}$ such that for every $A \in \mathscr{A}$ we have $|X \cap A| \leqslant \aleph_0$. We put $B = {}^\omega 2 \setminus \bigcup \mathscr{A}$ and $b = [B]_{\mathscr{L}}$. It is easy to see that for every Borel set $C \subseteq B$ we have ($C \in \mathscr{L}$ iff $C \in \mathscr{J}$). From this we conclude that $\mathscr{B}/\mathscr{J} \cong (\mathscr{B}/\mathscr{L})_b$; hence

$$\Delta(\mathscr{B}/\mathscr{J}) = \Delta((\mathscr{B}/\mathscr{L})_b) = \Delta(\mathscr{B}/\mathscr{L}) = \Delta(\mathscr{L}).$$

In our model, ${}^\omega 2$ is not a union of less than $\aleph_2$ sets from $\mathscr{L}$. This fact implies that $\Delta(\mathscr{L}) = \aleph_2$.

(2) Let us consider ${}^\kappa 2$ for a given infinite cardinal number κ. Let $\mathscr{B}_\kappa$ be the σ-algebra generated by $\{I_\xi : \xi < \kappa\}$, where $I_\xi = \{f \in {}^\kappa 2 : f(\xi) = 0\}$. A slight modification of the proof of Theorem 1.1 gives us the following result.

THEOREM 3.2. *Suppose* $\mathscr{I}$ *is a σ-ideal on* ${}^\omega 2$ *with a base contained in* $\mathscr{B}_\kappa$ *such that* $\mathscr{B}_\kappa/\mathscr{I}$ *is σ-saturated. Then* $\Delta(\mathscr{B}_\kappa/\mathscr{I}) \leqslant \Delta(\mathscr{I}) \cdot \aleph_0$.

Let us consider the product measure on ${}^\kappa 2$ arising from the measure on $\{0,1\}$ which gives values 0.5 to 0 and 1. Let $\mathscr{L}_\kappa$ be the ideal of zero measure subsets of ${}^\kappa 2$. Let $e_\xi \in {}^\kappa 2$ be a function defined by $e_\xi(\xi) = 1$ and $e_\xi(\eta) = 0$ for $\eta \neq \xi$. For $B \in \mathscr{B}_\kappa$ we define $\operatorname{supp}(B) = \{\xi \in \kappa : B + e_\xi \neq B\}$ (we treat ${}^\kappa 2$ here as a product of κ copies of the cyclic group $\{0,1\}$).

Note that for any $B \in \mathscr{B}_\kappa$ we have $|\operatorname{supp}(B)| \leqslant \aleph_0$. For any finite function σ, the domain of which is contained in κ and range in $\{0,1\}$, let $\langle\sigma\rangle = \{f \in {}^\kappa 2 : \sigma \subseteq f\}$. Now suppose $A \in \mathscr{B}_\kappa$. We define $A^* = A \setminus \bigcup\{\langle\sigma\rangle : \langle\sigma\rangle \cap A \in \mathscr{L}_\kappa \wedge$ domain of σ is contained in $\operatorname{supp}(A)\}$.

Note that for every $A \in \mathscr{B}_\kappa$ we have $[A]_{\mathscr{L}_\kappa} = [A^*]_{\mathscr{L}_\kappa}$, and for every open set U, if $A \cap U \neq \varnothing$, then $A \cap U \notin \mathscr{L}_\kappa$.

We can now imitate the proof of Lemma 2.2 to obtain the following inequality.

LEMMA 3.3. $\Delta(\mathscr{B}_\kappa \setminus \mathscr{L}_\kappa, \subseteq) \leqslant \Delta(\mathscr{B}_\kappa/\mathscr{L}_\kappa)$.

Suppose $A \in \mathscr{B}_\kappa \setminus \mathscr{L}_\kappa^c$. Then the set $\bigcap\{A + f : f \in {}^\kappa 2 \wedge f^{-1}(\{1\})$ is a finite subset of $\operatorname{supp}(A)\}$ belongs to $\mathscr{L}_\kappa$. This observation allows us to generalize the proof of Lemma 2.3:

LEMMA 3.4. $\Delta(\mathscr{L}_\kappa) \leqslant \Delta(\mathscr{B}_\kappa \setminus \mathscr{L}_\kappa)$.

From the above lemmas and theorem we obtain the following equality.

COROLLARY 3.5. *For any infinite cardinal number* κ, $\Delta(\mathscr{L}_\kappa) = \Delta(\mathscr{B}_\kappa/\mathscr{L}_\kappa)$.

(3) A precise estimation of $\Delta(\mathscr{L})$ is impossible. Clearly $\aleph_1 \leqslant \Delta(\mathscr{L}) \leqslant 2^{\aleph_0}$. Since the additivity of Lebesgue measure is less than or equal to $\Delta(\mathscr{L})$, Martin's Axiom implies that $\Delta(\mathscr{L}) = 2^{\aleph_0}$. The theory ZFC + $2^{\aleph_0} + \aleph_2 + \Delta(\mathscr{L}) = \aleph_1$ is also (relatively) consistent (see [**1**]).

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