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A LOCAL CHARACTERIZATION OF NOETHERIAN AND DEDEKIND RINGS

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ABSTRACT. Let R be a ring and M a maximal ideal of R. Then R is Noetherian if and only if every ideal contained in M is finitely generated; R is Dedekind if and only if every nonzero ideal contained in M is invertible.

Let R be a ring. It is well known that if the localizations R_P are Noetherian (respectively Dedekind) at all primes P, then R need not be Noetherian (respectively Dedekind). In this note, we will show that the properties of being Noetherian or Dedekind for R have nevertheless a local nature.

PROPOSITION 1. *Let R be a ring and M a maximal ideal of R. Then,*

(a) *R is Noetherian if (and only if) every ideal contained in M is finitely generated.*

(b) *R is Dedekind if (and only if) every nonzero ideal contained in M is invertible.*

(c) *There exists an integer n such that every ideal of R is generated by n elements if (and only if) there exists an integer m such that every ideal of R contained in M is generated by m elements.*

If I is an ideal of R, $\mu(I)$ will denote the minimal number of elements that are needed to generate I. Proposition 1 is clearly a consequence of the following:

PROPOSITION 2. *Let R be a ring and M a maximal ideal of R. Let I be an ideal of R.*

(a) *If M is finitely generated, then I is finitely generated if and only if $I \cap M$ is finitely generated.*

(b) *If M is invertible, then I is invertible if and only if $I \cap M$ is invertible.*

(c) $\mu(I) \leqslant 1 + \mu(I \cap M)$.

The proof of this proposition will rely on the following:

LEMMA. *Let R be a ring. Let I, J be two ideals of R such that $I + J$ is invertible. Then*

$$\mu(I) - \mu(I + J) \leqslant \mu(I \cap J) \leqslant \mu(I) + \mu(J).$$

PROOF. Consider the map $\varphi\colon I \oplus J \to R$ defined by $\varphi(a, b) = a + b$. The image of φ is equal to $I + J$ which is invertible by hypothesis, hence projective; thus, $I \oplus J \simeq (I + J) \oplus \ker\varphi$. Now,

$$\ker\varphi = \{(a, b) \in I \oplus J / a + b = 0\} = \{(a, -a) / a \in I \cap J\}$$

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is isomorphic to $I \cap J$. Thus, $I \oplus J \simeq (I + J) \oplus (I \cap J)$. From this, we get that $\mu(I) \leqslant \mu(I \oplus J) = \mu((I + J) \oplus (I \cap J)) \leqslant \mu(I + J) + \mu(I \cap J)$, hence that $\mu(I) - \mu(I + J) \leqslant \mu(I \cap J)$. We also get that

$$\mu(I \cap J) \leqslant \mu((I + J) \oplus (I \cap J)) = \mu(I \oplus J) \leqslant \mu(I) + \mu(J).$$

PROOF OF PROPOSITION 2. (a) If $I \subseteq M$, we have $\mu(I) = \mu(I \cap M)$. If $I \not\subseteq M$, then $I + M = R$ and, by the lemma, we have $\mu(I) - 1 \leqslant \mu(I \cap M) \leqslant \mu(I) + \mu(M)$.

(b) If $I \subseteq M$, there is nothing to do. If $I \not\subseteq M$, then I and M are comaximal and therefore $I \cap M = IM$; M being invertible, we have that $I \cap M$ is invertible if and only if I is invertible.

(c) Immediate consequence of the lemma.

REMARK 1. In the same spirit as a theorem of I. Cohen [**1**, Theorem 2, p. 29], we could ask the following: If R is a ring, if M is a maximal ideal of R and if every prime ideal contained in M is finitely generated, then is R Noetherian? The answer is no as the following example shows.

Let (A, p) be a one-dimensional nondiscrete valuation ring, and let $R = A[X]$. Since p is the only nonzero prime ideal of A, there exists a maximal ideal M of $A[X]$ such that $M \cap A = (0)$. By [**2**, Theorem 2.7, p. 384], M is a principal ideal. Since the height of M is clearly equal to 1, we have that every prime ideal of $A[X]$ contained in M is principal; nevertheless, $A[X]$ is not Noetherian since A is not Noetherian.

REMARK 2. Proposition 1 and Proposition 2 can be easily generalized taking the intersection of a finite number of maximal ideals instead of just one. As a corollary, one obtains that if R is a quasi-local ring, then R is Noetherian (respectively Dedekind) if and only if every nonzero ideal contained in the Jacobson radical of R is finitely generated (respectively invertible).

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