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A ZETA-FUNCTION ASSOCIATED WITH ZERO TERNARY FORMS[1]

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ABSTRACT. Consider the zeta-function associated with zero ternary forms defined as

$$\tilde{\xi}(t)=\sum_{x}\frac{1}{|\det x|^{t}}\qquad(\operatorname{Re} t\geqslant 2),$$

where x runs over all $\mathrm{SL}_3(\mathbf{Z})$-inequivalent zero ternary forms. We shall approximate $\tilde{\xi}(t)$ by another zeta-function which we can compute explicitly. By the approximation, we see that $\tilde{\xi}(2)$ is very close to $2\zeta(2)\zeta(2)$ which gives the contribution of zero ternary forms to the dimension formula of Siegel's cusp forms of degree three (computing via Selberg Trace Formula) up to a constant multiple.

1. Introduction. For each pair of nonzero integers s_{13} and s_2, we define $\Delta(s_{13}, s_2)$ to be the set of ternary forms

$$\begin{bmatrix} 0 & 0 & s_{13} \\ 0 & s_2 & s_{23} \\ s_{13} & s_{23} & s_3 \end{bmatrix}.$$

Let

$$\mathscr{P}=\left\{U=\begin{bmatrix} 1 & u & v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix}\middle|\, u, v, w \text{ integers}\right\}.$$

$\mathscr{P}$ operates on $\Delta(s_{13}, s_2)$ by the action $S \to {}^tUSU$. Let $\mu(s_{13}, s_2)$ be the number of inequivalent representatives of $\Delta(s_{13}, s_2)$ under the operation of $\mathscr{P}$. Consider the zeta function $\xi(t)$ defined as

$$\xi(t)=\sum_{s_2\neq 0}\sum_{s_{13}=1}^{\infty}\frac{\mu(s_{13}, s_2)}{\left(s_2 s_{13}^2\right)^{t}}. \tag{1}$$

We shall prove

THEOREM A. *For* $\operatorname{Re} t \geqslant 2$, *we have*

$$\xi(t)=\left\{\frac{5}{2}+\frac{3}{2}\left(\frac{2^{-2t+1}+2^{-3t}-2^{-5t+2}}{1-2^{-3t}}\right)\right\}\frac{\zeta(t)\zeta(2t-1)\zeta(3t-1)}{\zeta(3t)}, \tag{2}$$

where $\zeta(t)$ *is the Riemann zeta-function.*

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2. The special case $t = 2$.

LEMMA 1. *Let $\delta(s_{13}, s_2)$ be the subset of $\Delta(s_{13}, s_2)$, defined by*

$$\begin{cases} 0 \leqslant s_{23} < (s_{13}, s_2) = g.c.d. \text{ of } s_{13} \text{ and } s_2, \\ 0 \leqslant s_3 < l(s_{23}), \end{cases}$$

where $l(s_{23})$ is the least positive integer in the set

$$G = \left\{ 2k \cdot \frac{s_{23}s_{13}}{(s_{13}, s_2)} + k^2 \cdot \frac{s_{13}^2 s_2}{(s_{13}, s_2)^2} + 2ns_{13} \middle| k, n \in \mathbf{Z} \right\}.$$

Then we have

(1) *for each $S \in \Delta(s_{13}, s_2)$, there exists $U \in \mathscr{P}$ such that ${}^t USU \in \delta(s_{13}, s_2)$; and*
(2) *if $S_1, S_2 \in \delta(s_{13}, s_2)$, $U \in \mathscr{P}$ and $S_1 = {}^t US_2U$, then $S_1 = S_2$.*

PROOF. For

$$U = \begin{bmatrix} 1 & m & n \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & s_{13} \\ 0 & s_2 & s_{23} \\ s_{13} & s_{23} & s_3 \end{bmatrix}$$

in $\mathscr{P}$ and $\Delta(s_{13}, s_2)$, respectively, we let

$${}^t USU = \begin{bmatrix} 0 & 0 & s_{13} \\ 0 & s_2 & s'_{23} \\ s_{13} & s'_{23} & s'_3 \end{bmatrix}.$$

Then a simple calculation shows

$$s'_{23} = s_{23} + ms_{13} + ps_2,$$
$$s'_3 = s_3 + 2ps_{23} + p^2 s_2 + 2ns_{13}.$$

First we choose integers m, p so that $0 \leqslant s'_{23} < (s_{13}, s_2)$. Note that the integral solutions of the equation $ms_{13} + ps_2 = 0$ are given by

$$p = \frac{ks_{13}}{(s_{13}, s_2)}, \quad m = \frac{-ks_2}{(s_{13}, s_2)}, \qquad k \text{ an integer.}$$

Substituting the value of p as above into s'_3, we get

$$s'_3 = s_3 + 2k \cdot \frac{s_{23}s_{13}}{(s_{13}, s_2)} + k^2 \cdot \frac{s_{13}^2 s_2}{(s_{13}, s_2)^2} + 2ns_{13}.$$

When k and n range over all integers, the set G is a principal ideal of $\mathbf{Z}$. Hence we can choose s' as asserted. (2) is obvious.

REMARK 1. The set $k^2 \cdot s_{13}^2 s_2/(s_{13}, s_2)^2$ is a multiple of s_{13}. If we let $\tilde{s}_{13} = s_{13}/(s_{13}, s_2)$ and $\tilde{s}_2 = s_2/(s_{13}, s_2)$, then we have

$$l(s_{23}) = \begin{cases} 2(s_{23}\tilde{s}_{13}, s_{13}) & \text{if } \tilde{s}_2 \text{ is even and } s_{23} \neq 0, \\ (2s_{23}\tilde{s}_{13}, s_{13}) & \text{if } \tilde{s}_2 \text{ is odd and } s_{23} \neq 0, \\ 2|s_{13}| & \text{if } s_{23} = 0. \end{cases}$$

REMARK 2. The set $\delta(s_{13}, s_2)$ in Lemma 1 is a subset of $M(s_{13}, s_2)$ which consists of matrices in $\Delta(s_{13}, s_2)$ with $0 \leqslant s_{23} < (s_{13}, s_2)$ and $0 \leqslant s_3 < 2s_{13}$. However, $\delta(s_{13}, s_2) \neq M(s_{13}, s_2)$ in general as shown by the following example:

$$ {}^t\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}. $$

From this lemma, it is easy to see that $\mu(s_{13}, s_2) = \mu(s_{13}, -s_2) = \mu(-s_{13}, s_2) = \mu(-s_{13}, -s_2)$. Hence it suffices to consider the case when s_{13} and s_2 are positive integers. Here are some particular values of $\mu(s_{13}, s_2)$ for $s_{13} = 1, 2, 3, 4, 5$.

$$ \mu(1, s_2) = \begin{cases} 2 & \text{if } s_2 \text{ is even}, \\ 1 & \text{if } s_2 \text{ is odd}; \end{cases} $$

$$ \mu(2, s_2) = \begin{cases} 4 & \text{if } s_2 = 4n+1, 4n+3, \\ 6 & \text{if } s_2 = 4n+2, 4n; \end{cases} $$

$$ \mu(3, s_2) = \begin{cases} 3 & \text{if } s_2 = 6n+1, 6n+5, \\ 6 & \text{if } s_2 = 6n+2, 6n+4, \\ 5 & \text{if } s_2 = 6n+3, \\ 10 & \text{if } s_2 = 6n; \end{cases} $$

$$ \mu(4, s_2) = \begin{cases} 8 & \text{if } s_2 \text{ is odd}, \\ 12 & \text{if } s_2 = 8n+2, 8n+6, \\ 16 & \text{otherwise}; \end{cases} $$

$$ \mu(5, s_2) = \begin{cases} 5 & \text{if } s_2 = 10n+1, 10n+3, 10n+7, 10n+9, \\ 10 & \text{if } s_2 = 10n+2, 10n+4, 10n+6, 10n+8, \\ 9 & \text{if } s_2 = 10n+5, \\ 18 & \text{if } s_2 = 10n. \end{cases} $$

For each fixed positive integer s_{13}, we define

$$ \eta(s_{13}) = \sum_{s_2 \neq 0} \frac{\mu(s_{13}, s_2)}{s_{13}^4 s_2^2} \cdot \frac{1}{2\zeta(2)}. \tag{3} $$

LEMMA 2. *For any positive integer k, we have*

$$ \eta(2^k) = \frac{2^{k+1}}{2^{4k}} + \frac{2^k}{2^{4k+2}} + \cdots + \frac{2^k}{2^{6k}}. \tag{4} $$

PROOF. Since the values of $\mu(s_{13}, s_2)$ are computed via Lemma 1 as

$$ \mu(2^k, s_2) = \begin{cases} 2^{k+1} + m \cdot 2^k & \text{if } (s_2, 2^{k+1}) = 2^m, 0 \leqslant m \leqslant k, \\ 2^{k+1} + k \cdot 2^k & \text{if } (s_2, 2^{k+1}) = 2^{k+1}, \end{cases} $$

it follows that

$$ \begin{aligned} \eta(2^k) &= \sum_{s_2=1}^{\infty} \frac{\mu(2^k, s_2)}{2^{4k} s_2^2} \cdot \frac{1}{\zeta(2)} \\ &= \frac{2^{k+1}}{2^{4k}} + \frac{2^k}{2^{4k+2}} + \cdots + \frac{2^k}{2^{6k}}. \end{aligned} $$

LEMMA 3. *If p is an odd prime and m is a positive integer, then*

$$\eta(p^m) = \frac{5}{4}\left[\frac{p^m}{p^{4m}} + \frac{\phi(p^m)}{p^{4m+2}} + \cdots + \frac{\phi(p^m)}{p^{6m}}\right], \tag{5}$$

where $\phi(p^m) = p^{m-1}(p-1)$ is the Euler ϕ-function.

PROOF. Let $1 \leqslant n \leqslant m$ and $(s_2, 2p^m) = 2p^n$. If $l(s_{23})$ is the integer defined in Lemma 1 for such s_2, then we have $l(0) = 2p^m$ and $l(kp^u) = 2p^{m-n+u}$ if $(k, p) = 1$ and u is a nonnegative integer. (The total numbers of such k's is $\phi(p^{n-u})$.) Hence we get

$$\mu(p^m, s_2) = 2[p^m + n\phi(p^m)].$$

For the case $(s_2, 2p^m) = p^n$, we get $\mu(p^m, s_2) = p^m + n\phi(p^m)$ in the same manner. Hence our lemma follows from the definition of $\eta(p^m)$.

THEOREM B.

$$\xi(2) = \frac{65}{24} \cdot \frac{\zeta(2)\zeta(3)\zeta(5)}{\zeta(6)}.$$

PROOF. By the definition of η and Lemma 1, we have

$$\eta(a) = \alpha \sum_{d|a} \frac{a\phi(d)}{a^4 d^3},$$

where $\alpha = 2$ if a is even and $\alpha = \frac{5}{4}$ if a is odd. Consequently, if m and n are relatively prime integers, then a direct calculation shows $\eta(mn) = \frac{4}{5}\eta(m)\eta(n)$.

Let $\tilde{\eta}(a) = \frac{4}{5}\eta(a)$. Then by the previous lemmas, we have the following properties for $\tilde{\eta}(m)$:

$$\tilde{\eta}(2^k) = \frac{8}{5}\left[\frac{2^k}{2^{4k}} + \frac{2^{k-1}}{2^{4k+2}} + \cdots + \frac{2^{k-1}}{2^{6k}}\right], \tag{1}$$

$$\tilde{\eta}(p^k) = \frac{p^k}{p^{4k}} + \frac{\phi(p^k)}{p^{4k+2}} + \cdots + \frac{\phi(p^k)}{p^{6k}} \tag{2}$$

if p is an odd prime,

(3) $\tilde{\eta}(mn) = \tilde{\eta}(m)\tilde{\eta}(n)$ if m and n are relative prime integers.

Hence

$$\sum_{m=1}^{\infty} \tilde{\eta}(m) = \prod_{p:\text{ prime}} (1 + \tilde{\eta}(p) + \tilde{\eta}(p^2) + \cdots + \tilde{\eta}(p^n) + \cdots).$$

For odd prime p, we have

$$1 + \sum_{k=1}^{\infty} \tilde{\eta}(p^k) = \frac{(1-p^{-6})}{(1-p^{-3})(1-p^{-5})}.$$

For the special case $p = 2$, we have

$$1 + \sum_{k=1}^{\infty} \tilde{\eta}(2^k) = \frac{13}{12} \cdot \frac{(1-2^{-6})}{(1-2^{-3})(1-2^{-5})}.$$

Hence

$$\xi(2) = 2\zeta(2)\cdot \sum_{m=1}^{\infty} \eta(m) = \frac{5}{2}\zeta(2) \sum_{m=1}^{\infty} \tilde{\eta}(m)$$
$$= \frac{65}{24}\zeta(2) \prod_{p:\ \text{prime}} \frac{(1-p^{-6})}{(1-p^{-3})(1-p^{-5})}$$
$$= \frac{65}{24}\cdot\frac{\zeta(2)\zeta(3)\zeta(5)}{\zeta(6)}.$$

3. The general case. For each fixed positive integer s_{13}, we define

$$\eta_t(s_{13}) = \frac{1}{2\zeta(2t)}\cdot \sum_{s_2\neq 0} \frac{\mu(s_{13}, s_2)}{\left(s_{13}^4 s_2^2\right)^t}, \qquad \operatorname{Re} t \geqslant 1. \tag{6}$$

Then we have

(1) $\eta_t(1) = \frac{5}{4}$,

(2) $$\eta_t(2^k) = 2\left(\frac{2^k}{2^{2kt}} + \frac{2^{k-1}}{2^{(2k+2)t}} + \cdots + \frac{2^{k-1}}{2^{3kt}}\right),$$

(3) $$\eta_t(p^k) = \frac{5}{4}\left(\frac{p^k}{p^{2kt}} + \frac{\phi(p^k)}{p^{(2k+2)t}} + \cdots + \frac{\phi(p^k)}{p^{3kt}}\right)$$

if p is an odd prime,

(4) $\eta_t(mn) = \frac{4}{5}\eta_t(m)\eta_t(n)$ if m and n are relative prime integers.

From the computation we carried out before, we get

THEOREM A. *For* $\operatorname{Re} t \geqslant 2$, *we have*

$$\xi(t) = \left[\frac{5}{2} + \frac{3}{2}\cdot\frac{2^{-2t+1} + 2^{-3t} - 2^{-5t+2}}{1-2^{-3t}}\right]\cdot\frac{\zeta(t)\zeta(2t-1)\zeta(3t-1)}{\zeta(3t)}.$$

4. Application and remark. Let S be a 3×3 integral symmetric matrix of rank 3. We call S a zero ternary form if S represents zero in rational integers; i.e. there exists a nonzero integral vector $u = [u_1, u_2, u_3]$ such that $uSu^t = 0$. Hence there exists a unimodular integral matrix U such that [3]

$$USU^t = \begin{bmatrix} 0 & 0 & s_{13} \\ 0 & s_2 & s_{23} \\ s_{13} & s_{23} & s_3 \end{bmatrix}.$$

Set

$\mathscr{A}$: the set of representatives of zero ternary forms under the operation of unimodular matrices of $GL_3(\mathbf{Z})$ by the action $S \to USU^t$,

$\mathscr{B}$: the set of representatives of

$$G = \bigcup_{s_2\neq 0} \bigcup_{s_{13}=1}^{\infty} \Delta(s_{13}, s_2)$$

under the operation of $\mathscr{P}$.

Then for each $S \in \mathscr{A}$, there exists a unimodular U in $\mathrm{GL}_3(\mathbf{Z})$ such that $USU^t \in \mathscr{B}$. Hence we can approximate the series

$$\tilde{\xi}(t) = \sum_{x \in \mathscr{A}} \frac{1}{|\det x|^t}, \qquad \operatorname{Re} t \geqslant 2,$$

by the series

$$\xi(t) = \sum_{x \in \mathscr{B}} \frac{1}{|\det x|^t} \qquad (\operatorname{Re} t \geqslant 2)$$
$$= \sum_{s_2 \neq 0} \sum_{s_{13}=1}^{\infty} \frac{\mu(s_{13}, s_2)}{(s_{13}^2 s_2)^t}$$

which contains $\tilde{\xi}(t)$ as a subseries. If we use the approximate values of zeta-functions as

$$\zeta(2) = 1.6449341, \qquad \zeta(3) = 1.2020569,$$
$$\zeta(5) = 1.0369297, \qquad \zeta(6) = 1.0173431,$$

it follows that

$$\frac{65}{48} \cdot \frac{\zeta(3)\zeta(5)}{\zeta(6)} \simeq 1.0086268\zeta(2).$$

Hence it is possible that $\tilde{\xi}(2) = 2\zeta(2)\zeta(2)$ (a formula which is hard to verify directly).

Note that the zeta-function $\tilde{\xi}(t)$ we defined here is a constant multiple of a subseries of $\xi_2(s, L)$ appearing in [2] (restricted L to zero ternary forms). This tells us that a constant multiple (the constant is $2^{-6}\pi^{-4}$ by a direct computation from the Selberg Trace Formula) of $\tilde{\xi}(2)$ gives the contribution of ternary forms to the dimension formula of Siegel's cusp forms of degree three with respect to $\mathrm{Sp}(3, \mathbf{Z})$.

References

1. T. Shintani, *Zeta-functions associated with the vector space*, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **22** (1975), 25–65.

2. C. L. Siegel, *Lectures on quadratic forms*, Tata Inst. of Fundamental Research, Bombay, 1957.

3. ______, *Über die Zetafunktionen indefiniter quadratischer Formen*, Math. Z. **43** (1938), 682–708.

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