A CHARACTERIZATION IN THE SPACE OF CONVOLUTION OPERATORS

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ABSTRACT. We give a characterization of C^∞ elements in the space of convolution operators θ'_c , which belong to the Schwartz space $\mathcal{S}$.

In the space of convolution operators θ'_c , also referred to as the space of distributions which are rapidly decreasing at ∞ , there are C^∞ elements which are not rapidly decreasing in the sense of the Schwartz space $\mathcal{S}$. For a preliminary discussion of the spaces θ'_c and its image θ_M under the Fourier transform, see Treves [5, pp. 314–321], or Schwartz [4, Chapter VII, §8].

The purpose of this note is to give a necessary and sufficient condition for C^∞ elements in θ'_c to belong to $\mathcal{S}$. Notation used here but not defined is standard; see Hörmander [1, 2].

Let $\rho(\xi) \in \mathcal{S}^m$ be any “constant coefficient” symbol, that is, $\rho(\xi) \in C^\infty(\mathbf{R}^n)$ and satisfies for each multi-index α the estimate

$$(1) \quad |D_\xi^\alpha \rho(\xi)| \leq c_\alpha (1 + |\xi|)^{m-|\alpha|}, \quad \xi \in \mathbf{R}^n,$$

where c_α are constants. Let $\rho(D)$ be the pseudo-differential operator corresponding to the symbol $\rho(\xi)$. Then clearly, for $v \in \mathcal{E}'$, one has

$$\widehat{\rho(D)v}(\xi) = \rho(\xi) \hat{v}(\xi) \in \theta_M.$$

Thus $\rho(D)$ may be considered as a map $\rho(D): \mathcal{E}' \rightarrow \theta'_c$ and is sometimes referred to as a *Friedrichs operator*. With this notation our main result becomes the following

THEOREM. *Let $f \in \theta'_c \cap C^\infty(\mathbf{R}^n)$. Then $f \in \mathcal{S}$ if and only if there exists a Friedrichs operator $\rho(D)$ such that $f = \rho(D)v$ for some distribution v with compact support.*

REMARK. A weaker form of the Theorem, namely that if $\chi(\xi)$ is a C^∞ function on $\mathbf{R}^n$, positively homogeneous of degree 0 for $|\xi| > 1$ and if $v \in \mathcal{E}'$ is such that $\chi(D)v$ is C^∞ , then $|\chi(\xi)\hat{v}(\xi)| = O(|\xi|^{-N})$ for all positive integers N , has been mentioned in Nirenberg [3, p. 42].

The proof of the Theorem is based on the following

LEMMA. $\{\rho(D)v: v \in \mathcal{E}'\} \cap C^\infty(\mathbf{R}^n) \subset \mathcal{S}$.

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PROOF. Suppose $\rho(D)v \in C^\infty(\mathbf{R}^n)$ for some $v \in \mathcal{E}'$, $v \neq 0$. Then we may write (see [2, Theorem 2.13, p. 149])

$$(2) \quad \rho(D) = \rho_1 + \rho_2$$

where ρ_1 is properly supported, that is, maps C_0^∞ into itself and $\mathcal{E}'$ into itself; ρ_2 has a C^∞ kernel and defines a map on $\mathcal{E}' \rightarrow C^\infty(\mathbf{R}^n)$. This decomposition is defined by a decomposition of the Schwartz kernel K_ρ associated with $\rho(D)$, in the form

$$(3) \quad K_{\rho_1} = \theta K_\rho \quad \text{and} \quad K_{\rho_2} = (1 - \theta) K_\rho$$

where $\theta \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ which takes values in $[0, 1]$, equals 1 in a neighbourhood of the diagonal of $\mathbf{R}^n \times \mathbf{R}^n$ and is *properly supported*.

The expression for the kernel K_ρ is given by

$$(4) \quad K_\rho(x, y) = \frac{(2\pi)^{-n}}{(x-y)^\alpha} \int_{\mathbf{R}^n} e^{i(x-y) \cdot \xi} (-D_\xi)^\alpha \rho(\xi) d\xi, \quad x, y \in \mathbf{R}^n, x \neq y,$$

where the multi-index α is arbitrary but chosen so that the integral converges absolutely. Now by (2), we have

$$(5) \quad \rho(D)v = \rho_1(v) + \rho_2(v) \Rightarrow \rho_1(v) \in C_0^\infty(\mathbf{R}^n).$$

Our aim now is to prove that $\rho_2(v) \in \mathcal{S}$. To this end, we shall first establish the form of the C^∞ function $\rho_2(v)$. If we choose $f \in C_0^\infty(\mathbf{R}^n)$ such that $f = 1$ in a neighbourhood of $\text{supp}(v)$, then (see [2, Definition 2.11, p. 148])

$$(6) \quad \rho_2(v)(x) = \rho_2(fv)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix \cdot \xi} p_f(x, \xi) \hat{v}(\xi) d\xi$$

where p_f stands for a symbol of ρ_2 in $\mathcal{S}^{-\infty}$ and is given by the expression

$$(7) \quad e^{ix \cdot \xi} p_f(x, \xi) = \rho_2(fe^{i\langle \cdot, \xi \rangle})(x)$$

where $e^{i\langle \cdot, \xi \rangle}$ denotes for each ξ the function $e^{ix \cdot \xi}$. Now expressing the action of ρ_2 in terms of its kernel, we have from (3) and (4)

$$(8) \quad \rho_2(fe^{i\langle \cdot, \xi \rangle})(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \left[\frac{(1-\theta)}{(x-y)^\alpha} \int_{\mathbf{R}^n} e^{i(x-y) \cdot t} (-D_t)^\alpha \rho(t) dt \right] f(y) e^{iy \cdot \xi} dy.$$

Noting that $\theta(x, y)f(y)$ has compact support since θ is properly supported and integrating by parts with respect to y , we obtain, for arbitrary multi-indices β_1, β_2

$$(9) \quad \sup_{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n} |\xi^{\beta_1} D_x^{\beta_2} \rho_2(fe^{i\langle \cdot, \xi \rangle})(x)| < \infty$$

when α is chosen so that $|\alpha| > (m + n + |\beta_1| + |\beta_2|)$. Now from (6) and (7) we have

$$(10) \quad \rho_2(v)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \hat{v}(\xi) \rho_2(fe^{i\langle \cdot, \xi \rangle})(x) d\xi$$

where we note that $\hat{v}$ has polynomial growth, say of order N , and the integral is absolutely convergent because of (9). Differentiating under the integral sign in (10), it is clear that for arbitrary multi-indices μ, ν we have

$$\sup_{x \in \mathbf{R}^n} |x^\mu D_x^\nu \rho_2(v)(x)| < \infty$$

when the multi-index α appearing in (8) is chosen large enough, that is,

$$|\alpha| > \max\{|\mu|, m + 2n + N + 1 + |\nu|\}.$$

Thus, $\rho_2(v) \in \mathcal{S}$. Hence by (5) $\rho(D)v \in \mathcal{S}$ and the proof of the Lemma is complete.

PROOF OF THE THEOREM. $f \in \mathcal{S} \Leftrightarrow \hat{f}(\xi) \in \mathcal{S} \Leftrightarrow \hat{f}(\xi) \in \mathcal{S}^m \forall$ real m . Also we have $f = \hat{f}(D)\delta$, where δ is the Dirac measure on $\mathbf{R}^n$. The rest of the proof follows from the Lemma.

REFERENCES

1. L. Hörmander, *Linear partial differential operators*, Grundlehren Math. Wiss., Vol. 116, Springer-Verlag, Berlin and New York, 1963.
2. ———, *Pseudo-differential operators and hypoelliptic equations*, Singular Integrals, Proc. Sympos. Pure Math., vol. 10, Amer. Math. Soc., Providence, R. I., 1967.
3. L. Nirenberg, *Lectures on linear partial differential equations*, CBMS Regional Conf. Ser. in Math., no. 17, Amer. Math. Soc., Providence, R. I., 1972.
4. L. Schwartz, *Théorie des distributions*, 2nd ed., Hermann, Paris, 1966.
5. F. Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

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