

ON AN L_1 -APPROXIMATION PROBLEM

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ABSTRACT. Let $C_w[a, b]$ denote the space of real continuous functions with norm $\|f\|_w = \int_a^b |f(x)| w(x) dx$, where w is a positive bounded weight. It is known that if a subspace $M_n \subset C_w[a, b]$ satisfies a certain A -property, then M_n is a Chebyshev subspace of $C_w[a, b]$ for all w . We prove that the A -property is also necessary for M_n to be Chebyshev in $C_w[a, b]$ for each w .

Let W be the set of all measurable bounded weights w on $[a, b]$ such that $\inf\{w(x): x \in [a, b]\} > 0$. Consider the space $L_w[a, b]$ of real-valued Lebesgue integrable functions with norm

$$(1) \quad \|f\|_w = \int_a^b |f(x)| w(x) dx,$$

where $w \in W$. By the famous theorem of Krein, there are no finite-dimensional Chebyshev subspaces in $L_w[a, b]$ —for any space $U_n \subset L_w[a, b]$, $\dim U_n = n$, there exists $f \in L_w[a, b]$ having nonunique best approximation in U_n . The situation is different if we restrict our attention to $C_w[a, b]$, the space of continuous functions with norm (1). By the classical result of Jackson and Krein, if U_n is a Haar space on (a, b) , then it is a Chebyshev subspace of $C_w[a, b]$ for any $w \in W$. (Recall that an n -dimensional space of continuous functions on $[a, b]$ is said to be Haar on (a, b) if its elements have at most $n - 1$ zeros at (a, b) .) Havinson [2] gave a partial converse of this statement, proving that if U_n is a Chebyshev subspace of $C_w[a, b]$ for any $w \in W$, and no nontrivial element of U_n vanishes on an interval, then U_n is a Haar space on (a, b) . The assumption that elements of the subspace do not vanish on intervals is essential in Havinson's theorem, since, as proved in recent years, different families of splines with fixed nodes are also Chebyshev in $C_w[a, b]$ (see [1, 7, 5]). It turned out that Haar spaces and spline functions have a common property which is crucial for uniqueness of the L_1 -approximation (see Strauss [6], Sommer [5] and Kroó [3]). Let us give the corresponding definition. Given a space U_n of continuous functions on $[a, b]$, we set $U_n^* = \{u^*: u^* \text{ is continuous on } [a, b] \text{ and there exists } u \in U_n \text{ such that } |u^*| = |u| \text{ on } [a, b]\}$. Then we say that U_n satisfies the A -property (or is an A -space) if for any $u^* \in U_n^* \setminus \{0\}$ there exists $u \in U_n \setminus \{0\}$ such that $u = 0$ a.e. on $Z(u^*)$ and $uu^* \geq 0$ on $[a, b] \setminus Z(u^*)$. Here, and in what

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follows, $Z(f) = \{x \in [a, b]: f(x) = 0\}$. Obviously, Haar spaces satisfy the A -property. Moreover, it is known [4–6] that different families of spline functions are also A -spaces. Furthermore, by a theorem of Strauss [6], A -spaces are Chebyshev in $C_w[a, b]$ for any $w \in W$. This raises the natural question of whether or not the A -property is also necessary in order for a subspace to be Chebyshev in $C_w[a, b]$ for each $w \in W$. In this paper we give a positive answer to this problem.

THEOREM. *Let U_n be a finite-dimensional subspace of $C_w[a, b]$. Then in order for U_n to be a Chebyshev subspace of $C_w[a, b]$ for each $w \in W$, it is necessary and sufficient that U_n satisfy the A -property.*

We also show that Havinson's result can be easily deduced from the above theorem.

Note that the A -property is not necessary for uniqueness with respect to a given $w \in W$; see e.g. [5].

PROOF OF THE THEOREM. As mentioned above, the sufficiency of the A -property was verified by Strauss [6] (also see [3] for a more general setting).

Let us prove the necessity part of the Theorem.

By a theorem of Strauss [8] $U_n \subset C_w[a, b]$ is a Chebyshev subspace of $C_w[a, b]$ for a given $w \in W$ if and only if for any $u^* \in U_n^* \setminus \{0\}$ there exists $u \in U_n$ such that

$$(2) \quad \left| \int_{N(u^*)} w u \operatorname{sign} u^* \right| > \int_{Z(u^*)} w |u|,$$

where $N(u^*) = [a, b] \setminus Z(u^*)$. Assume now that U_n is Chebyshev in $C_w[a, b]$ for any $w \in W$ and consider an arbitrary $u^* \in U_n^* \setminus \{0\}$. Then by Strauss's result, for any $w \in W$ there exists $u \in U_n$ such that (2) holds.

Set $\tilde{U}_k = \{g \in U_n: g = 0 \text{ a.e. on } Z(u^*)\}$. Evidently, \tilde{U}_k is a linear subspace of U_n of some dimension $1 \leq k \leq n$, because there exists a function $u \in U_n$ such that

$$(3) \quad |u| = |u^*|, \quad x \in [a, b].$$

Now we state that there exists a $g_0 \in \tilde{U}_k$ such that, for any $w \in W$,

$$(4) \quad \int_{N(u^*)} w g_0 \operatorname{sign} u^* \neq 0.$$

Assume that, on the contrary, for any $g \in \tilde{U}_k$, we can find a $w \in W$ satisfying

$$(5) \quad \int_{N(u^*)} w g \operatorname{sign} u^* = 0.$$

Let $\{g_1, \dots, g_k\}$ be a basis in \tilde{U}_k . Then, by (5), for any $\{b_i\}_{i=1}^k \in \mathbf{R}^k$, there exists $w \in W$ for which

$$(6) \quad 0 = \int_{N(u^*)} w \left(\sum_{i=1}^k b_i g_i \right) \operatorname{sign} u^* = \sum_{i=1}^k b_i \int_{N(u^*)} w g_i \operatorname{sign} u^*.$$

Set $A_0 = \{(\int_{N(u^*)} w g_i \operatorname{sign} u^*)_{i=1}^k: w \in W\}$. Evidently, A_0 is a convex subset of \mathbf{R}^k . Furthermore, (6) implies that A_0 has nonempty intersection with any hyperplane $H_d = \{c \in \mathbf{R}^k: \langle d, c \rangle = 0\}$, where $d \in \mathbf{R}^k \setminus \{0\}$ is arbitrary and $\langle \cdot, \cdot \rangle$ denotes the

inner product in \mathbf{R}^k . Moreover, we claim that A_0 is open. Consider an arbitrary point $C \in A_0$. Then for some $w_c \in W$,

$$(7) \quad C = \{C_i\}_{i=1}^k = \left\{ \int_{N(u^*)} w_c g_i \operatorname{sign} u^* \right\}_{i=1}^k.$$

Obviously, the functions $g_i \operatorname{sign} u^*$, $1 \leq i \leq k$, are linearly independent on $(a, b) \setminus Z(u^*)$. Therefore, there exist k distinct points $x_1, \dots, x_k \in (a, b) \setminus Z(u^*)$ such that the vectors $l_j = \{g_i(x_j) \operatorname{sign} u^*(x_j)\}_{i=1}^k$, $1 \leq j \leq k$, are also linearly independent in \mathbf{R}^k . Since the functions $g_i \operatorname{sign} u^*$, $1 \leq i \leq k$, are continuous on the open set $(a, b) \setminus Z(u^*)$, it follows that, for $h > 0$ sufficiently small, the vectors

$$l_j^* = \left\{ \int_{x_j-h}^{x_j+h} g_i \operatorname{sign} u^* \right\}_{i=1}^k \in \mathbf{R}^k \quad (1 \leq j \leq k)$$

will also be linearly independent. (We assume that $[x_j - h, x_j + h] \subset (a, b) \setminus Z(u^*)$, $1 \leq j \leq k$.) Let $\inf_{x \in [a, b]} w_c(x) = \Theta > 0$ and set, for each $1 \leq j \leq k$,

$$(8) \quad w_j(x) = \begin{cases} 0, & x \in [a, b] \setminus [x_j - h, x_j + h], \\ \Theta/2, & x \in [x_j - h, x_j + h]. \end{cases}$$

Then, evidently, $w_c + w_j$ and $w_c - w_j$ belong to W , $1 \leq j \leq k$. Therefore, setting

$$\alpha_j = \left\{ \int_{N(u^*)} (w_c + w_j) g_i \operatorname{sign} u^* \right\}_{i=1}^k, \quad \beta_j = \left\{ \int_{N(u^*)} (w_c - w_j) g_i \operatorname{sign} u^* \right\}_{i=1}^k$$

for each $1 \leq j \leq k$, we obtain that $\alpha_j, \beta_j \in A_0$ ($1 \leq j \leq k$), and, by (7) and (8),

$$(9) \quad \alpha_j - C = C - \beta_j = (\Theta/2) l_j^* \quad (1 \leq j \leq k).$$

Since $\alpha_j, \beta_j \in A_0$ for each $1 \leq j \leq k$, we have that, by (9) and convexity of A_0 ,

$$A_0 \ni \sum_{j=1}^k r_j \alpha_j + \sum_{j=1}^k \tau_j \beta_j = C + \frac{\Theta}{2} \sum_{j=1}^k (r_j - \tau_j) l_j^*$$

for any $r_j, \tau_j \geq 0$ such that $\sum_{j=1}^k (r_j + \tau_j) = 1$. This and the linear independence of vectors l_j^* , $1 \leq j \leq k$, imply that A_0 contains a k -dimensional ball with center at C . Hence, A_0 is open.

Now let us show that $0 \in A_0$. Indeed, if $0 \notin A_0$, then evidently 0 belongs to the boundary of A_0 . Furthermore, given the convex set A_0 and the point 0 at its boundary, we can find a hyperplane H_d such that $\langle d, c \rangle \geq 0$ for every $c \in A_0$ ($d \in \mathbf{R}^k$). On the other hand, A_0 should have nonempty intersection with H_d , i.e., for some $c^* \in A_0$, $\langle d, c^* \rangle = 0$. Moreover, A_0 is open, hence there exists a ball S with center at c^* belonging to A_0 . Then $\langle d, c \rangle \geq 0$ for every $c \in S$ and, therefore, $\langle d, c^* \rangle > 0$, a contradiction. By this contradiction, we obtain that $0 \in A_0$, i.e., for some $\tilde{w} \in W$,

$$(10) \quad \int_{N(u^*)} \tilde{w} g_i \operatorname{sign} u^* = 0, \quad 1 \leq i \leq k.$$

Let now U_n be spanned by $\{g_i\}_{i=1}^n$, where, as above, $\{g_i\}_{i=1}^k$ is a basis in \tilde{U}_k . Set $U'_{n-k} = \text{span}\{g_{k+1}, \dots, g_n\}$. Note that $k < n$, otherwise (10) would contradict our initial assumption that for each $w \in W$ there exists $u \in U_n$ such that (2) holds. Since $\int_{Z(u^*)} |\tilde{g}| > 0$ for any $\tilde{g} \in U'_{n-k} \setminus \{0\}$, for some $\gamma_0 > 0$ we have

$$(11) \quad \int_{Z(u^*)} |\tilde{g}| \geq \gamma_0 \max_{x \in [a, b]} |\tilde{g}(x)| \quad (\tilde{g} \in U'_{n-k}).$$

Consider now a weight w^* equal to \tilde{w} on $N(u^*)$ and to $((b-a)/\gamma_0) \sup_{x \in [a, b]} \tilde{w}(x)$ on $Z(u^*)$. Then for any $g = g^* + \tilde{g} \in U_n$, where $g^* \in \tilde{U}_k$ and $\tilde{g} \in U'_{n-k}$, we have, by (10) and (11),

$$\begin{aligned} \left| \int_{N(u^*)} w^* g \operatorname{sign} u^* \right| &= \left| \int_{N(u^*)} \tilde{w} \tilde{g} \operatorname{sign} u^* \right| \\ &\leq (b-a) \sup_{x \in [a, b]} \tilde{w}(x) \max_{x \in [a, b]} |\tilde{g}(x)| \\ &\leq \frac{b-a}{\gamma_0} \sup_{x \in [a, b]} \tilde{w}(x) \int_{Z(u^*)} |\tilde{g}| = \int_{Z(u^*)} w^* |\tilde{g}|. \end{aligned}$$

This again contradicts our assumption that, for any $w \in W$, there exists a $u \in U_n$ such that (2) holds.

By this contradiction we obtain that for some $g_0 \in \tilde{U}_k$, (4) should be true for any $w \in W$. This implies that $g_0 u^*$ does not change its sign on $N(u^*)$. Indeed, if $(-1)^i g_0(x_i) u^*(x_i) > 0$ ($i = 1, 2$) for some $x_1, x_2 \in N(u^*)$, then, also, $(-1)^i g_0 \operatorname{sign} u^* > 0$ in a neighborhood of x_i ($i = 1, 2$) belonging to $N(u^*)$. Then for appropriately chosen $w_1, w_2 \in W$, we shall have

$$(-1)^i \int_{N(u^*)} w_i g_0 \operatorname{sign} u^* > 0, \quad i = 1, 2.$$

This and convexity of W imply that for some $\bar{w} \in W$, $\int_{N(u^*)} \bar{w} g_0 \operatorname{sign} u^* = 0$, contradicting (4). Therefore, we may assume that $g_0 u^* \geq 0$ on $N(u^*) = [a, b] \setminus Z(u^*)$. Furthermore, since $g_0 \in \tilde{U}_k$, it follows that $g_0 = 0$ a.e. on $Z(u^*)$. This proves the A -property of U_n . The theorem is proved.

The next proposition shows that Havinson's result is a consequence of the above theorem.

PROPOSITION. *Let U_n be an A -space on $[a, b]$ and assume that elements of U_n do not vanish on intervals. Then U_n is a Haar space on (a, b) .*

PROOF. Let $M(n; n+1) = \mathbf{R}^{n(n+1)}$ be the space of all $n \times (n+1)$ matrices. Denote by $M^*(n; n+1)$ the set of $n \times (n+1)$ matrices all whose $n \times n$ determinants are nonzero. It can easily be shown that $M^*(n; n+1)$ is dense in $M(n; n+1) = \mathbf{R}^{n(n+1)}$ (see e.g. [2]).

Assume now that U_n is not Haar, i.e., some $g_0 \in U_n \setminus \{0\}$ has n distinct zeros $a < x_1 < x_2 < \dots < x_n < b$. Set $I_i = [x_i, x_{i+1})$ ($0 \leq i \leq n$; $x_0 = a$; $x_{n+1} = b$) and consider the matrix

$$B_{n, n+1}(w) = \left(\int_{I_j} w g_i \right)_{\substack{1 \leq i \leq n \\ 0 \leq j \leq n}} \in M(n; n+1),$$

where $w \in W$ and $\{g_i\}_{i=1}^n$ is a basis in U_n . Furthermore, set $B_{n,n+1}^* = \{B_{n,n+1}(w) : w \in W\}$. We state that $B_{n,n+1}^* \cap M^*(n; n+1) = \emptyset$. Assume, to the contrary, that for some $w \in W$, $B_{n,n+1}(w) \in M^*(n; n+1)$. Then the system of linear equations

$$(12) \quad \sum_{j=0}^n a_j \int_{I_j} w g_i = 0, \quad 1 \leq i \leq n,$$

has nonvanishing solutions $a_j \neq 0$, $0 \leq j \leq n$. Consider now the weight $w^* \in W$ equal to $|a_j|w$ on I_j ($0 \leq j \leq n$) and set $u^* = |g_0| \text{sign } a_j$ on I_j ($0 \leq j \leq n$). Then, evidently, $u^* \in U_n^*$, $Z(u^*) = Z(g_0)$, and, by (12), for each $g \in U_n$,

$$(13) \quad \left| \int_{[a,b] \setminus Z(u^*)} w^* \frac{u^*}{|g_0|} g \right| \leq \int_{Z(u^*)} w^* |g|.$$

On the other hand, by the A -property, there exists $u \in U_n \setminus \{0\}$ such that $u = 0$ a.e. on $Z(u^*)$ and $uu^* \geq 0$ on $[a, b] \setminus Z(u^*)$, which contradicts (13).

Thus $B_{n,n+1}^* \cap M^*(n; n+1) = \emptyset$. Using the fact that $M^*(n; n+1)$ is dense in $\mathbf{R}^{n(n+1)}$, we obtain that $B_{n,n+1}^*$ should be nowhere dense in $\mathbf{R}^{n(n+1)}$, but $B_{n,n+1}^*$ is a convex subset of $\mathbf{R}^{n(n+1)}$, and 0 belongs to its boundary. Hence, $B_{n,n+1}^*$ should belong to a hyperplane, i.e., for some $C_{i,j} \in \mathbf{R}$ ($1 \leq i \leq n$, $0 \leq j \leq n$) such that $\sum_{i,j} |C_{i,j}| > 0$, we obtain that for any $w \in W$,

$$(14) \quad \sum_{j=0}^n \sum_{i=1}^n C_{i,j} \int_{I_j} w g_i = \sum_{j=0}^n \int_{I_j} w h_j = 0,$$

where $h_j = \sum_{i=1}^n C_{i,j} g_i \in U_n$, $0 \leq j \leq n$. Furthermore, it can be easily derived from (14) that $h_j = 0$ on I_j , while at least one of the h_j 's should not be identically zero on $[a, b]$. This contradicts our assumption that no element of U_n vanishes on an interval. The proposition is verified.

Now by the Theorem and the Proposition, we obtain the following

COROLLARY (HAVINSON [2]). *If U_n is a Chebyshev subspace of $C_w[a, b]$ for any $w \in W$ and no nontrivial element of U_n vanishes on an interval, then U_n is a Haar space on (a, b) .*

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