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# ON LIPSCHITZ FUNCTIONS OF NORMAL OPERATORS

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ABSTRACT. It is shown that if $N$ and $M$ are normal operators on a separable, complex Hilbert space $H$, and $f$ is a Lipschitz function on $\Omega = \sigma(N) \cup \sigma(M)$ (i.e., $|f(z) - f(w)| \leqslant k|z - w|$ for some positive constant $k$ and all $z, w \in \Omega$), then $\|f(N)X - Xf(M)\|_2 \leqslant k\|NX - XM\|_2$ for any operator $X$ on $H$. In particular, $\|f(N) - f(M)\|_2 \leqslant k\|N - M\|_2$.

Let $H$ denote a separable, complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators acting on $H$. An operator $T \in B(H)$ is said to belong to the Hilbert-Schmidt class $C_2$ in case $\sum_{i,j}|(Te_i, e_j)|^2 = \sum_i\|Te_i\|^2$ is finite for some (hence, for all) complete orthonormal systems $\{e_i\}$ in $H$. For $T \in C_2$, let $\|T\|_2 = (\sum_i\|Te_i\|^2)^{1/2}$ be the Hilbert-Schmidt norm of $T$. The properties of Hilbert-Schmidt operators are described in Schatten **[9]** and Gohberg-Krein **[7]**.

In their work on Scattering theory, W. O. Amrein and D. B. Pearson proved **[1**, Theorem 2] that if $A$ is a selfadjoint operator with pure continuous spectrum and $f$ is a Lipschitz function on $\sigma(A)$ (the spectrum of $A$) i.e., $|f(t) - f(s)| \leqslant k|t - s|$, then $\|f(A)X - Xf(A)\|_2 \leqslant k\|AX - XA\|_2$ for all $X \in C_2$. Utilizing Voiculescu's perturbation property of normal operators **[10]**, we now establish the following considerable generalization of the Amrein-Pearson result.

THEOREM. *Let $N$ be a normal operator. Let $f$ be a function defined on $\Omega = \sigma(N)$. If $|f(z) - f(w)| \leqslant k|z - w|$ for all $z, w \in \Omega$ and some positive constant $k$, then*

$$\|f(N)X - Xf(N)\|_2 \leqslant k\|NX - XN\|_2 \quad \text{for all } X \in B(H).$$

PROOF. Given $\varepsilon > 0$, let $N = D_\varepsilon + K_\varepsilon$, where $D_\varepsilon$ is diagonal and $\|K_\varepsilon\|_2 < \varepsilon$ **[10]**. If $D_\varepsilon e_n = \lambda_n e_n$ and $X = (x_{ij})$ is the corresponding matrix of $X$, relative to the basis $\{e_n\}$, then the $(i, j)$ entry for $D_\varepsilon X - XD_\varepsilon$ is $(\lambda_i - \lambda_j)x_{ij}$. Similarly the $(i, j)$ entry for $f(D_\varepsilon)X - Xf(D_\varepsilon)$ is $(f(\lambda_i) - f(\lambda_j))x_{ij}$. Since

$$\|D_\varepsilon X - XD_\varepsilon\|_2^2 = \sum_{i,j}\left|(\lambda_i - \lambda_j)x_{ij}\right|^2$$

and

$$\|f(D_\varepsilon)X - Xf(D_\varepsilon)\|_2^2 = \sum_{i,j}\left|\big(f(\lambda_i) - f(\lambda_j)\big)x_{ij}\right|^2,$$

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it follows that

$$(*) \qquad \|f(D_\varepsilon)X - Xf(D_\varepsilon)\|_2 \leqslant k\|D_\varepsilon X - XD_\varepsilon\|_2.$$

Next, let $N = \int_\Omega z\, dE(z)$ be the spectral representation of $N$. Then

$$\begin{aligned}\|((f(N) - f(D_\varepsilon))e_n\|^2 &= \int_\Omega |f(z) - f(\lambda_n)|^2\, d\|E(z)e_n\|^2 \\ &\leqslant k^2 \int_\Omega |z - \lambda_n|^2\, d\|E(z)e_n\|^2 \\ &= k^2\|(N - D_\varepsilon)e_n\|^2.\end{aligned}$$

Consequently, $\|f(N) - f(D_\varepsilon)\|_2 \leqslant k\|N - D_\varepsilon\|_2 = k\|K_\varepsilon\|_2 < \varepsilon k$. Thus $f(N) = f(D_\varepsilon) + C_\varepsilon$ with $\|C_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$. Since

$$\big|\|NX - XN\|_2 - \|D_\varepsilon X - XD_\varepsilon\|_2\big| \leqslant \|K_\varepsilon X - XK_\varepsilon\|_2 \leqslant 2\|K_\varepsilon\|_2\|X\| < 2\|X\|\varepsilon,$$

it follows that

$$\lim_{\varepsilon \to 0} \|D_\varepsilon X - XD_\varepsilon\|_2 = \|NX - XN\|_2.$$

Similarly we have $\lim_{\varepsilon \to 0}\|f(D_\varepsilon)X - Xf(D_\varepsilon)\|_2 = \|f(N)X - Xf(N)\|_2$. The required result now follows by letting $\varepsilon \to 0$ in $(*)$ above.

An alternative proof of the Theorem which was suggested by Professor Ando can be found in the author's doctoral thesis [**8**].

COROLLARY 1 (FUGLEDE'S THEOREM MODULO $C_2$ [**11**]). *Let $N$ be a normal operator. Then $\|NX - XN\|_2 = \|N^*X - XN^*\|_2$ for all $X \in B(H)$.*

PROOF. Apply the Theorem to the function $f(z) = \bar{z}$.

S. K. Berberian's trick allows us to extend the Theorem as follows.

COROLLARY 2. *Let $N$ and $M$ be normal operators and let $f$ be a function defined on the union of the spectra of $N$ and $M$. If $|f(z) - f(w)| \leqslant k|z - w|$ for all $z, w \in \sigma(N) \cup \sigma(M)$ and some positive constant $k$, then $\|f(N)X - Xf(M)\|_2 \leqslant k\|NX - XM\|_2$ for all $X \in B(H)$. In particular, $\|f(N) - f(M)\|_2 \leqslant k\|N - M\|_2$.*

PROOF. Define operators $L$ and $Y$ on the space $H \oplus H$ by

$$L = \begin{vmatrix} N & 0 \\ 0 & M \end{vmatrix}, \qquad Y = \begin{vmatrix} 0 & X \\ 0 & 0 \end{vmatrix}.$$

Then

$$\|NX - XM\|_2 = \|LY - YL\|_2$$

and

$$\|f(N)X - Xf(M)\|_2 = \|f(L)Y - Yf(L)\|_2.$$

Application of the Theorem to $L$ and $Y$ gives the required result.

The special case where $X = I$ in Corollary 2 is of particular interest in perturbation theory of linear operators (see [**4–6**]). Using the theory of Stieltjes double operator integrals, M. S. Birman and M. Z. Solomyak proved [**3**, Theorem 11] that if

$U$ and $V$ are unitary operators and $f$ is a function defined on the unit circle, whose derivative $f'$ is Lipschitz of order $\alpha > 0$, then $U - V \in C_2$ implies that $f(U) - f(V) \in C_2$. The special case where $f(z) = |z|$ in Corollary 2 above is also of great importance in the study of quasi-equivalence of quasi-free states of canonical commutation relations (see [2] and the references there).

For $T \in B(H)$, let the absolute value $|T|$ of $T$ be defined as $(T^*T)^{1/2}$. H. Araki and S. Yamagami proved [2, Theorem 1] that for any two operators $A$ and $B$ in $B(H)$, $\| |A| - |B| \|_2 \leqslant \sqrt{2} \|A - B\|_2$, and they remarked that $\sqrt{2}$ is the best possible coefficient for a general $A$ and $B$. However, if $A$ and $B$ are restricted to be selfadjoint, then the best coefficient is 1 instead of $\sqrt{2}$.

We conclude the paper with the following extension of the selfadjoint case.

COROLLARY 3. *Let $N$ and $M$ be normal operators. Then* $\| |N| - |M| \|_2 \leqslant \|N - M\|_2$.

PROOF. Apply Corollary 2 to the function $f(z) = |z|$.

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