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ON $|\bar{N}, p_n|_k$ SUMMABILITY FACTORS

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ABSTRACT. In this paper a theorem on $|\bar{N}, p_n|_k$ summability factors, which generalizes the theorem of Singh [5], has been proved.

1. Introduction. Let $\sum a_n$ be a given infinite series with partial sums s_n. By u_n^α we denote the nth Cesàro mean of order α ($\alpha > -1$) of the sequence (s_n). The series $\sum a_n$ is said to be summable $|C, \alpha|_k$ ($k \geqslant 1$), if

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty \qquad \text{(Flett [3])}.$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability is the same as $|C, 1|_k$ summability. Let (p_n) be a sequence of positive real constants such that $P_n = \sum_{v=0}^n p_v \to \infty$ as $n \to \infty$ ($P_{-1} = p_{-1} = 0$). The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \qquad (P_n \neq 0)$$

defines the sequence (t_n) of $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$ ($k \geqslant 1$), if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty \qquad \text{(Bor [1])}.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability. The series $\sum a_n$ is said to be bounded $[R, \log n, 1]_k$ ($k \geqslant 1$), if

$$\sum_{v=1}^{n} v^{-1} |s_v|^k = O(\log n) \quad \text{as } n \to \infty \text{ (Mishra [4])}.$$

The series $\sum a_n$ is said to be bounded $[\bar{N}, p_n]_k$ ($k \geqslant 1$), if

$$\sum_{v=1}^{n} p_v |s_v|^k = O(P_n) \quad \text{as } n \to \infty \text{ (Bor [2])}.$$

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It should be noted that, if we take $k = 1$ (resp. $p_n = 1/n$), then $[\bar{N}, p_n]_k$ boundedness is the same as $[\bar{N}, p_n]$ (resp. $[R, \log n, 1]_k$) boundedness.

2. Singh [5] proved the following theorem.

THEOREM A. *If Σa_n is bounded $[\bar{N}, p_n]$ and the sequences (λ_n) and (p_n) satisfy the following conditions*:

(i) $\Sigma_{n=1}^{m} p_n|\lambda n| = O(1)$,

(ii) $P_m|\Delta\lambda_m| = O(p_m|\lambda_m|)$, *then the series $\Sigma a_n P_n \lambda_n$ is summable $|\bar{N}, p_n|$.*

3. The object of this paper is to prove the following theorem.

THEOREM. *If Σa_n is bounded $[\bar{N}, p_n]_k$ and the sequences (λ_n) and (p_n) satisfy the same conditions in Theorem* A, *then the series $\Sigma a_n P_n \lambda_n$ is summable $|\bar{N}, p_n|_k$ $(k \geqslant 1)$.*

Note that, if we take $k = 1$ in our theorem, then we have Theorem A.

4. We shall require the following lemma for the proof of our theorem.

LEMMA. *If the sequences (λ_n) and (p_n) satisfy the conditions in Theorem* A, *then $P_m|\lambda_m| = O(1)$ as $m \to \infty$.*

PROOF. By Abel's partial summation formula, we have

$$\sum_{n=1}^{m} p_n\lambda_n = \sum_{n=1}^{m-1} P_n\Delta\lambda_n + P_m\lambda_m \Rightarrow |P_m\lambda_m| = \left|\sum_{n=1}^{m} p_n\lambda_n - \sum_{n=1}^{m-1} P_n\Delta\lambda_n\right|$$

$$\Rightarrow P_m|\lambda_m| < \sum_{n=1}^{m} p_n|\lambda_n| + \sum_{n=1}^{m-1} P_n|\Delta\lambda n|$$

$$= \sum_{n=1}^{m} p_n|\lambda_n| + O(1)\sum_{n=1}^{m-1} p_n|\lambda_n| = O(1).$$

Hence $P_m|\lambda_m| = O(1)$ as $m \to \infty$.

5. Proof of the theorem. Without any loss of generality we may assume that $a_0 = s_0 = 0$. Let T_n denote the $(\bar{N}, p_n)$ mean of the series $\Sigma a_n P_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r P_r \lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v P_v \lambda_v,$$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1} P_v a_v \lambda_v, \qquad n \geqslant 1.$$

Using Abel's transformation, we get

$$\begin{aligned}T_n - T_{n-1} &= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v P_v s_v \\ &\quad - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v p_{v+1} s_v \lambda_{v+1} + p_n s_n \lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.}\end{aligned}$$

To prove the theorem, by Minkowsi's inequality, it is sufficient to show that $\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}|T_{n,r}|^k < \infty$, for $r = 1, 2, 3, 4$. Now applying Hölder's inequality, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k \leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(P_v|\lambda_v|\right)^k p_v|s_v|^k \times \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1}.$$

Since

$$\left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1} = \left\{\frac{1}{P_{n-1}} \cdot P_{n-1}\right\}^{k-1} = O(1),$$

we have

$$\begin{aligned}\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(P_v|\lambda_v|\right)^k p_v|s_v|^k \\ &= O(1) \sum_{v=1}^{m} \left(P_v|\lambda_v|\right)^k p_v|s_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \left(P_v|\lambda_v|\right)^{k-1} p_v|\lambda_v|\, |s_v|^k = O(1) \sum_{v=1}^{m} p_v|s_v|^k|\lambda_v| \\ &= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| \sum_{r=1}^{v} p_r|s_r|^k + O(1)|\lambda_m| \sum_{v=1}^{m} p_v|s_v|^k \\ &= O(1) \sum_{v=1}^{m-1} P_v|\Delta\lambda_v| + O(1) P_m|\lambda_m| \\ &= O(1) \sum_{v=1}^{m-1} p_v|\lambda_v| + O(1) P_m|\lambda_m| = O(1) \quad \text{as } m \to \infty,\end{aligned}$$

by virture of the lemma and hypothesis. Since, $P_v|\Delta\lambda_v| = O(p_v|\lambda_v|)$, similarly we have

$$\begin{aligned}\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,2}|^k &= O(1) \sum_{v=1}^{m} p_v|s_v|^k|\lambda_v| \\ &= O(1) \sum_{v=1}^{m-1} p_v|\lambda_v| + O(1) P_m|\lambda_m| = O(1) \quad \text{as } m \to \infty,\end{aligned}$$

by virtue of the lemma and hypothesis. Again, similarly we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,3}|^k &\leqslant \sum_{n=2}^{m+1}\frac{p_n}{P_n(P_{n-1})^k}\left\{\sum_{v=1}^{n-1}P_v p_{v+1}|\lambda_{v+1}|\,|s_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(P_v|\lambda_v|\right)^k p_v|s_v|^k\times\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(P_v|\lambda_v|\right)^k p_v|s_v|^k \\
&= O(1)\sum_{v=1}^{m}\left(P_v|\lambda_v|\right)^k p_v|s_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\left(P_v|\lambda_v|\right)^k p_v|s_v|^k\frac{1}{P_v} \\
&= O(1)\sum_{v=1}^{m}\left(P_v|\lambda_v|\right)^{k-1} p_v|s_v|^k|\lambda_v| \\
&= O(1)\sum_{v=1}^{m}p_v|s_v|^k|\lambda_v| \\
&= O(1)\sum_{v=1}^{m-1}p_v|\lambda_v| + O(1)P_m|\lambda_m| = O(1) \quad \text{as } m\to\infty,
\end{aligned}$$

by virtue of the lemma and hypothesis. Finally, we have

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,4}|^k &= \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|p_ns_n\lambda_n|^k \\
&= \sum_{n=1}^{m}\left(P_n|\lambda_n|\right)^{k-1}p_n|\lambda_n|\,|s_n|^k = O(1)\sum_{n=1}^{m}p_n|s_n|^k\,|\lambda_n| = O(1) \quad \text{as } m\to\infty.
\end{aligned}$$

Therefore, we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,r}|^k = O(1) \quad \text{as } m\to\infty \text{ for } r = 1,2,3,4,$$

which completes the proof of the theorem.

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